The expected number of visibility events amongst uniformly distributed unit balls is linear

Olivier Devillers, Vida Dujmović, Hazel Everett,

Xavier Goaoc, Sylvain Lazard, Hyeon-Suk Na,

Sylvain Petitjean

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I. Problem

Amongst n unit balls

- whose centers are uniformly distributed in the universe $\mathcal{U}$ of radius R
- $\mu = \frac{n}{R^3}$ is constant (Why constant?)

what's the expected number of non-occluded (maximal) line segments tangent to 4 balls ($T4$ *segments*)?

II. Motivation: visibility problem

- Elementary query:
  Given two points p, q, are p and q mutually visible?

- Applications:
  Radiosity, Rendering (computer graphics),
  Hidden surface removal(computational geometry)

- Situation:
  Crucial, but not solved properly yet
  (slow, approximated and ad hoc solutions).

- Possible solution:
  Store *global visibility information* in a data structure
  which can then be efficiently queried
  (aspect graph, visibility graph, visibility complex).

III. Related works

Deterministic approach:

How many lines tangent to 4 amongst n unit balls whose centers are in general position?

- Lines:
  Upper bound: $O(n^{3+\epsilon})$ (Agarwal et al. '99)
  Lower bound: $\Omega(n^2)$ (Agarwal et al. '99)
- Non-occluded lines (T4 segments):
  Upper bound: $O(n^{3+\epsilon})$ (Agarwal et al. '99)
  Lower bound: $\Omega(n^2)$ (Lazard '01)

Probabilistic approach:

- F.Durand '99
  - Same model as ours, No occlusion
  - E(No. of lines tangent to 4 balls) $= O(n^{\frac{8}{3}})$.
- We show that
  - E(No. of T4 segments) $= O(n)$
    by taking $occlusion$ into account
  - Expected complexity of visibility complex is $O(n)$.

IV. Sketch of the proof

Let $\delta^{\omega}_{i,j,k,l}$ be the event that

- there exist the real line $L^{\omega}_{i,j,k,l}$ which is tangent to B_i, B_j, B_k, B_l at t_i, t_j, t_k, t_l in this order
- the segment $t_i t_l$ is not occluded (thus T4 segment).

Then

$$(\text{No. of T4 segments}) = \frac{1}{2} \sum_{(i,j,k,l)\in\mathcal{N}} \sum_{\omega=1}^{12} I_{\delta^{\omega}_{i,j,k,l}},$$

$$E(\textsf{No. of T4 segments}) = \frac{1}{2} \sum_{(i,j,k,l)\in\mathcal{N}} \sum_{\omega=1}^{12} \Pr(\delta^{\omega}_{i,j,k,l}).$$

$$
\begin{aligned}
\Pr\left(L^{\omega}_{i,j,k,l}\ \textsf{exists}\right) &\leq \Pr\left(p_j, p_k \in \mathcal{H}(p_i, p_l, 2) \cap \mathcal{U}\right) \\
&= \left(\frac{\textsf{Volume}(\mathcal{H}(p_i, p_l, 2) \cap \mathcal{U})}{\textsf{Volume}(\mathcal{U})}\right)^2
\end{aligned}
$$

$\Rightarrow$ Upper-Bound of Volume($\mathcal{H}(p_i, p_l, 2) \cap \mathcal{U}$)

$$\Pr\left(\delta^{\omega}_{i,j,k,l} \mid L^{\omega}_{i,j,k,l} \text{ exists}\right)$$

$$\leq \Pi_{\gamma \neq i,j,k,l} \Pr\left(p_\gamma \notin \mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}\right)$$

$$= \left(1 - \frac{\text{Volume}(\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U})}{\text{Volume}(\mathcal{U})}\right)^{n-4}$$

$\Rightarrow$ Lower-Bound of Volume$(\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U})$

Let $\mathbf{x}_{i,l}$ be the random variable representing the distance between p_i and p_l.

By the Total Probability Theorem,

$$\Pr(\delta^{\omega}_{i,j,k,l}) = \int_{x=0}^{2R} \Pr\left(x < \mathbf{x}_{i,l} \leq x + dx\right) \cdot \Pr\left(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x\right) \ .$$

Since

$$\Pr\left(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x\right) = \Pr\left(\delta^{\omega}_{i,j,k,l} \cap \text{Event } [L^{\omega}_{i,j,k,l} \text{ exists}] \mid \mathbf{x}_{i,l} = x\right),$$

$$\begin{aligned}\Pr\left(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x\right) = &\Pr\left(L^{\omega}_{i,j,k,l} \text{ exists} \mid \mathbf{x}_{i,l} = x\right) \\ &\cdot \Pr\left(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x, L^{\omega}_{i,j,k,l} \text{ exists}\right) \ .\end{aligned}$$

Thus we have

$$\Pr(\delta^{\omega}_{i,j,k,l}) = \int_{x=0}^{2R} \Pr\left(x < \mathbf{x}_{i,l} \leq x + dx\right) \cdot \Pr\left(L^{\omega}_{i,j,k,l} \text{ exists} \mid \mathbf{x}_{i,l} = x\right) \cdot \Pr\left(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x, L^{\omega}_{i,j,k,l} \text{ exists}\right) \ .$$

Calculation

- $\Pr\left(x < \mathbf{x}_{i,l} \leq x + dx\right) \leq \frac{3x^2}{R^3} \cdot dx$
- Intersecting Case: $B_i \cap B_l \neq \emptyset$
  - Volume($\mathcal{H}(p_i, p_l, 2) \cap \mathcal{U}$) $\leq (x - 2)$
  - Ignore occlusion

- Inside case: At least one of B_i and B_l is strictly inside $\mathcal{U}$
  - Volume($\mathcal{H}(p_i, p_l, 2) \cap \mathcal{U}$) $\leq (x-2)$
  - Volume($\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$) $\geq c \cdot x$

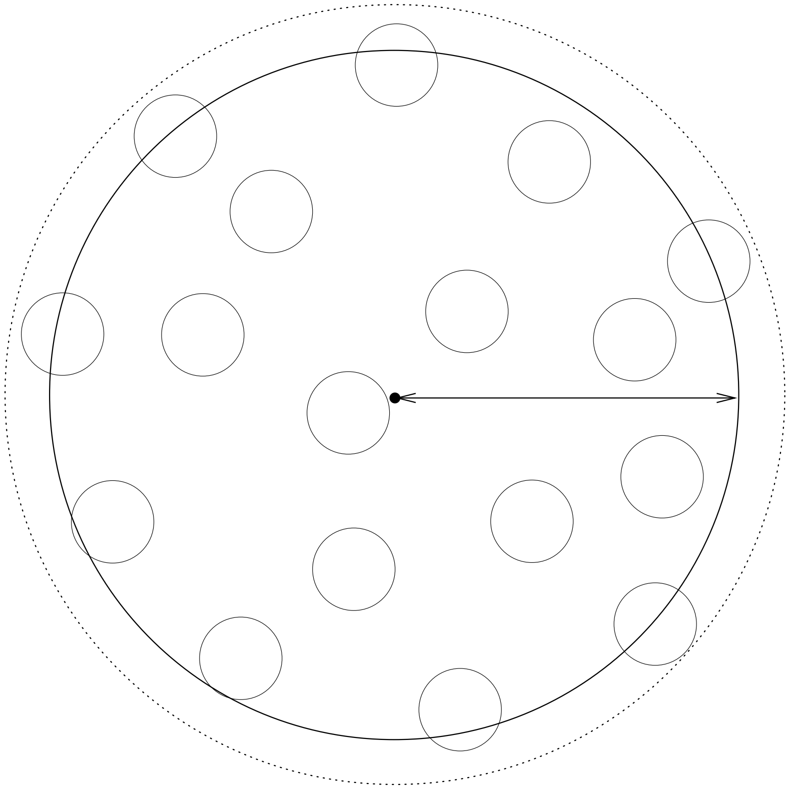
- Boundary case: $B_i, B_l \not\subset \mathcal{U}$

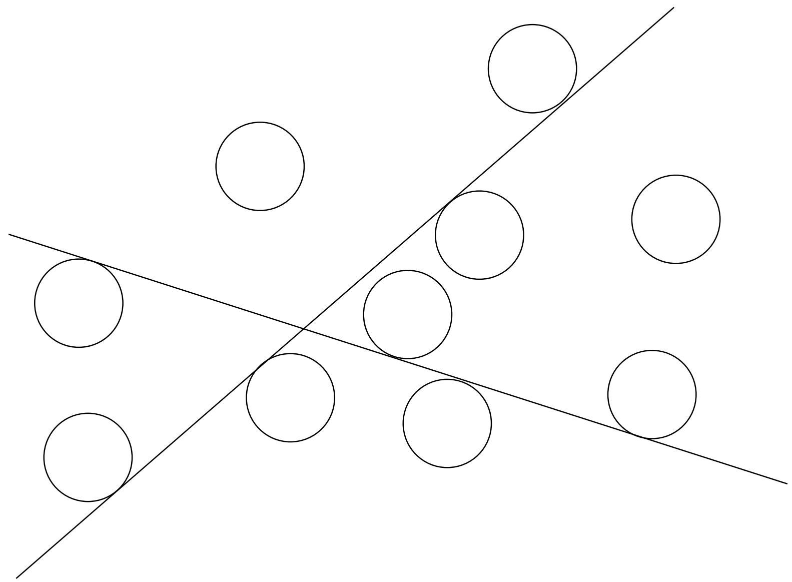
  Difficulty :

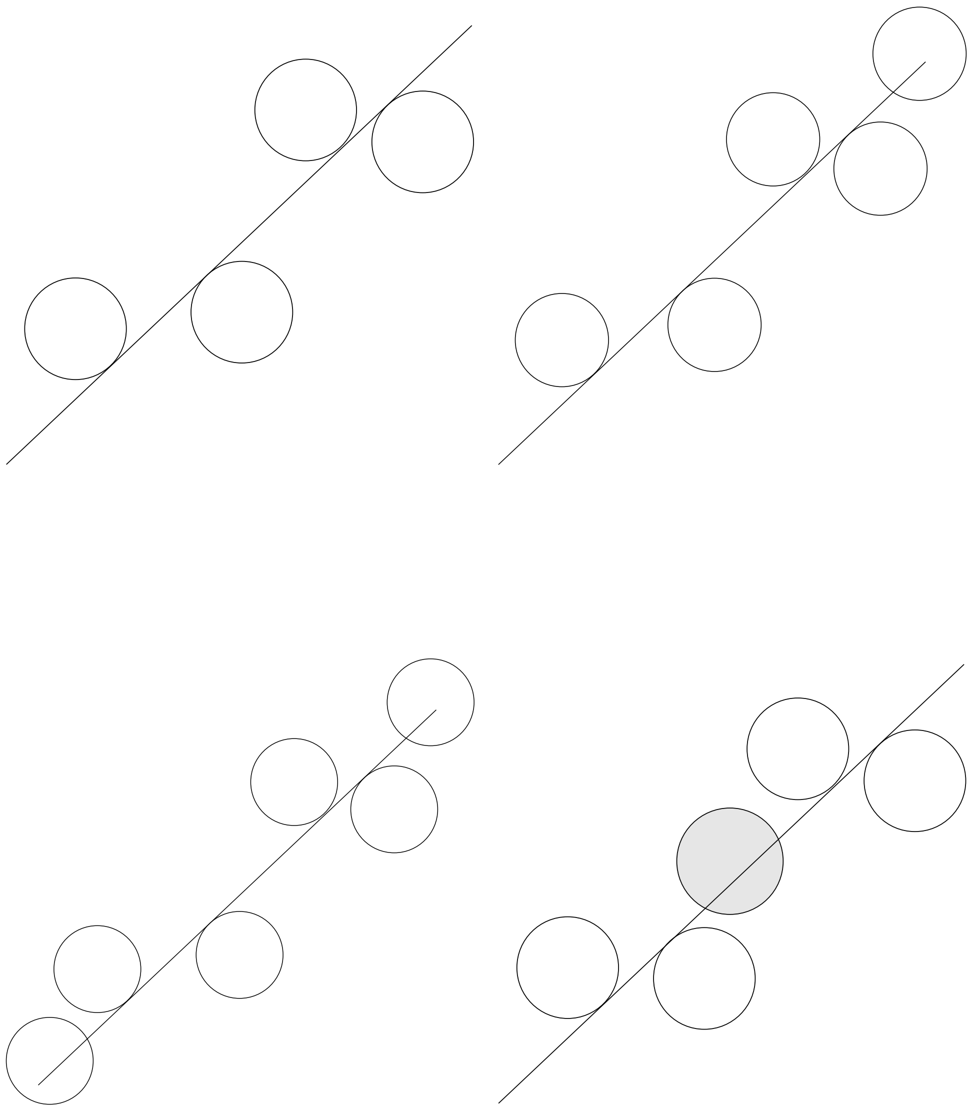
  – Volume($\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$) can be arbitrarily small, so Volume($\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$) $\not\geq c \cdot x$ as $y_i \to 0$ and $R \to \infty$

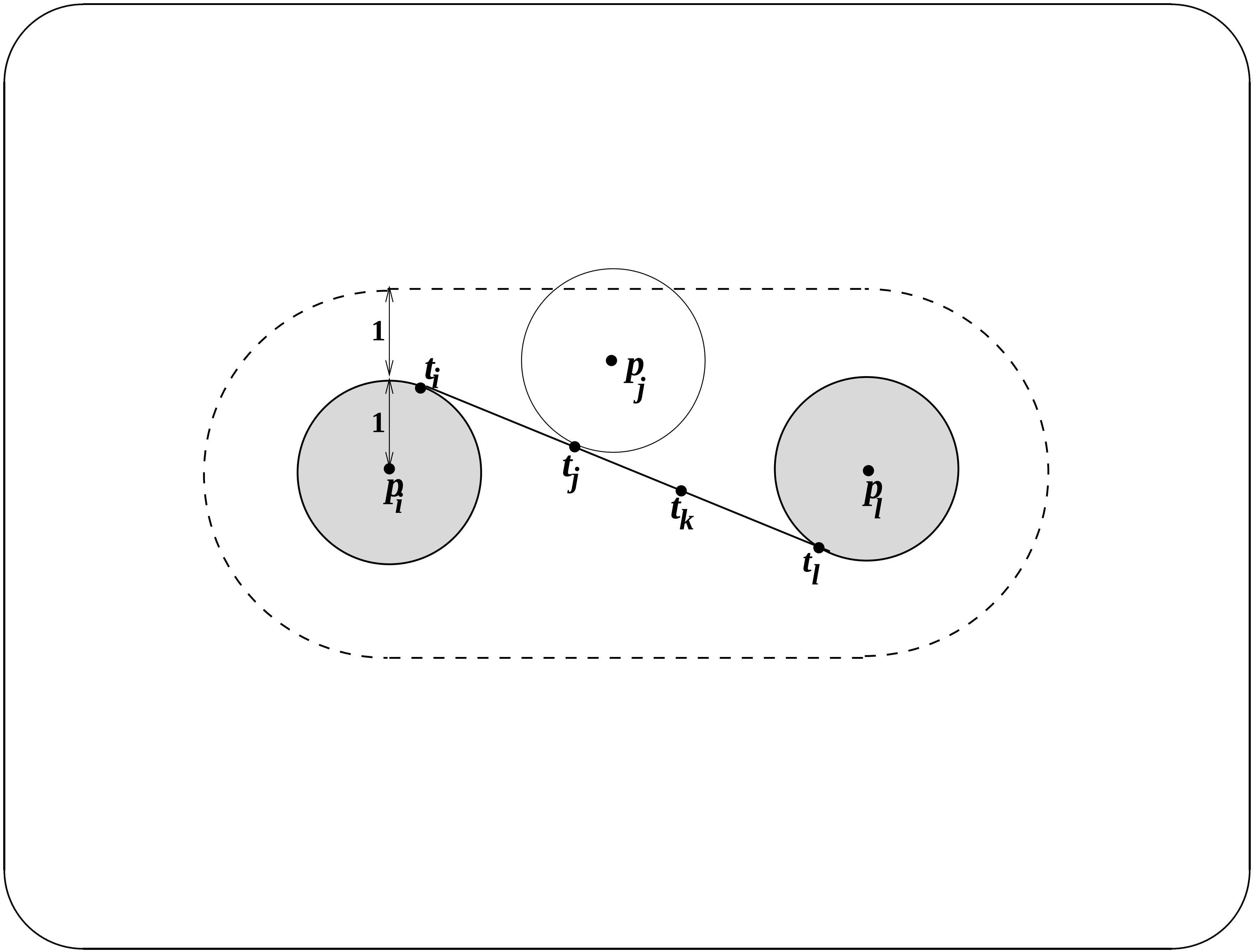
  – Not all tangents of $B_i, B_l \not\subset \mathcal{U}$ are "bad"

  $\Rightarrow$ Partition into "far tangents" and "near tangent"





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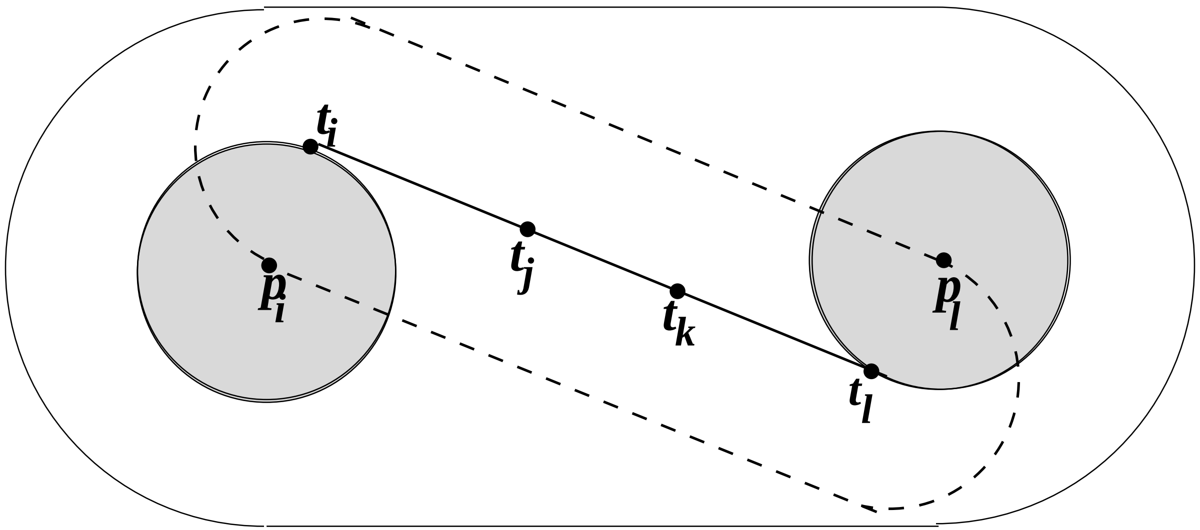

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