Optimality of the Parameterization of Quadrics and their Intersections

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Optimality

means here optimality in term of

the number of radicals involved

Quadrics of rank 1 or 2

Gauss-Jordan reduction leads to equations :

Rank 1 : x^2

Rational parameterization = no radicals

Clearly optimal

Rank 2 : $x^2 - ay^2$

Rational parameterization in $\mathbb{Q}(\sqrt{a})$

Optimal : There are regular rational points iff $\sqrt{a}$ is rational.

Quadrics of rank 3

Gauss-Jordan reduction leads to equation :

$$ax^2 + by^2 - cz^2$$

Parameterization :

$$x = 2uv,\ y = \frac{u^2 - abv^2}{b},\ z = \frac{u^2 + abv^2)}{\sqrt{bc}},\ t = w$$

$\Longrightarrow$ At most one square root

There exists a parameterization rational over $\mathbb{Q}$

$\Leftrightarrow$ **The conic** $ax^2 + by^2 - cz^2$ **has a rational point over** $\mathbb{Q}$

$\Leftrightarrow$ **Gauss-Jordan gives equation** $x^2 + y^2 - z^2$ **when starting from this point**

Example where a square root is needed :

$x^2 + y^2 - 3z^2$ has no rational point (Proof over $\mathbb{Z}/4\mathbb{Z}$)

Quadrics of signature (2,2)

Gauss-Jordan reduction leads to equation :

$$ax^2 + by^2 - cz^2 - dt^2$$

Parameterization :

$$x = \frac{uv' + au'v}{a},\; y = \frac{uv - bu'v'}{b},\; z = \frac{uv' - au'v}{\sqrt{ac}},\; t = \frac{uv + bu'v'}{\sqrt{bd}}$$

2 square roots

$\delta := abcd =$ discriminant of the quadric.

Invariant up to a square by change of coordinates

We claim :

$\sqrt{\delta}$ always needed

Another square root is needed iff the quadric has no rational point.

Quadrics of signature (2,2) (continued)

 Q admits a rational parameterization over $\mathbb{K}$, linear in one parameter

$\Leftrightarrow$ Q contains a line which is rational over $\mathbb{K}$

$\Leftrightarrow$ Q has a rational point over $\mathbb{K}$ and δ is a square in $\mathbb{K}$

$\Leftrightarrow$ Q has equation $x^2 + y^2 - z^2 - t^2$, in some rational frame over $\mathbb{K}$

$\Longrightarrow$ The field of a rational parameterization contains $\mathbb{Q}(\sqrt{\delta})$

Q has a rational point over $\mathbb{Q}\sqrt{\delta}$

$\Leftrightarrow$ It has a rational point over $\mathbb{Q}$ and equation $x^2 + y^2 - z^2 - \delta t^2$ in some rational frame.

$$x^2 + y^2 - 3z^2 - 11t^2$$

has no rational point over $\mathbb{Q}$

and no rational parameterization over $\mathbb{Q}(\sqrt{33})$

Proof over $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$

Intersections of quadrics

$\mathcal{P}$ and $\mathcal{Q}$ quadrics

The intersection is contained in any quadric of the pencil $\lambda\mathcal{P}+\mu\mathcal{Q}$

There is exactly one quadric of the pencil passing through a point outside of the intersection (linear equation to solve)

If the intersection is not empty nor singular

(i.e $\det(\lambda\mathcal{P}+\mu\mathcal{Q})$ has no multiple (projective) root),

the pencil contains a quadric of signature $(2,2)$ with a rational point

Parameterization through this quadric involves 2 $\sqrt{\ }$:

A constant one, $\sqrt{\delta}$, the discriminant of the quadric

One depending on the parameter, $\sqrt{\Delta}$

If the intersection is non singular, $\sqrt{\Delta}$ is needed

Putting the parameterization of the quadric in $\mathcal{P}$ gives an equation between the parameters.

Isomorphism of the intersection
with a curve in $\mathbb{P}_1 \times \mathbb{P}_1$, the spaces of parameters

De-homogenizing and re-homogenizing :
bi-rational equivalence with a curve in $\mathbb{P}_2$
of degree 4 with 2 ordinary singular points at infinity.

$\Longrightarrow$ Non singular intersection has genus 1

$\Longrightarrow$ No rational parameterization over $\mathbb{C}$

$\sqrt{\delta}$ is not needed iff a surface of degree 8 has a rational point

If $\sqrt{\delta}$ is not needed,

the intersection contains pair of conjugate points of degree 2

the quadric passing through a point on a rational line

defined by such a pair has a square discriminant

$\Leftrightarrow$ The surface

$$z^2 = \det(\mathcal{Q}(x,y,1,0)\,\mathcal{P} - \mathcal{P}(x,y,1,0)\,\mathcal{Q})$$

has a rational point.

This surface has no rational point in the case :

$$\mathcal{P} = 5y^2 + 6xy + 2z^2 - t^2 + 6zt;$$

$$\mathcal{Q} = 3x^2 + y^2 - z^2 - t^2$$

Singular intersections

0 is a multiple root of $\det(\mathcal{P} + \lambda Q)$

$\Leftrightarrow$ $\mathcal{P}$ has rank ≤ 2 (pair of plans) or

$\mathcal{P}$ is a cone and all quadrics of the pencil are tangent at its vertex

$\Leftrightarrow$ The intersection has a singular (complex) point which is also a singular point of $\mathcal{P}$

Up to a real projective transformation,

there are 28 singular intersections corresponding to 46 pencils.

Algorithm sketched by Sylvain Lazard

managed through the number of multiple roots of $\det(\lambda \mathcal{P} + \mu Q)$

$\Longrightarrow$

If the intersection does not consists in 4 lines

at most 2 $\sqrt{\ }$ for parameterizing all components

at most 1 $\sqrt{\ }$ may be unnecessary

it may be removed by finding a rational point on a conic

If the intersection consists in 4 non coplanar lines (skew quadrilateral)

at most 2 $\sqrt{\ }$ for each line

possibility of 3 $\sqrt{\ }$ for all lines together

always optimal for each line and for all lines together.

In the case of 4 concurrent lines

general degree 4 equation

Singular quartic

$\det(\lambda\mathcal{P} + \mu Q)$ has a single multiple root

Parameterize the cone corresponding to this root : 1 or 0 $\sqrt{\ }$

Plug it in the equation of $\mathcal{P}$

$\Longrightarrow$ equation which is linear in one of the parameter.

One $\sqrt{\ }$, zero if one finds a rational point on the cone

Cubic an line

$\det(\lambda\mathcal{P} + \mu Q)$ has 2 double roots, real or not

Parameterize through a $(2,2)$ quartic : no $\sqrt{\ }$

$\Longrightarrow$ Equation in the parameters factors (GCD computation) in
a univariate factor (the line)
and a factor linear in that variable (the cubic)

Totally rational parameterization

Two conics

$\det(\lambda\mathcal{P}+\mu Q)$ has 1 (rational) double root

The corresponding quadric is a pair of plans

Compute these plans : 1 or 0 $\sqrt{}$

Parameterize through a $(2,2)$ quartic and plug the parameters in the equations of the plans : 1 or 0 $\sqrt{}$

$\Longrightarrow$ Linear equations in the parameters

First $\sqrt{}$ is needed if the plans are not rational

Second $\sqrt{}$ may be avoided for each conic with a rational point on the field of the plans

Conic and 2 lines

$\det(\lambda\mathcal{P} + \mu Q)$ has 2 (rational) double roots

The corresponding quadrics are a cone and a pair of plans

Parameterize through a $(2,2)$ quadric : one $\sqrt{\ }$ iff the lines are irrational

$\Longrightarrow$ Equation in the parameters factors (GCD computation) in

2 univariate factors (the lines)

and a bilinear one (the conic)

At most 1 $\sqrt{\ }$

It may be avoideed for the conic iff it has a rational point

4 lines

$\det(\lambda\mathcal{P}+\mu Q)$ has 2 double roots

The corresponding quadrics are pairs of plans

If the roots are rational

Compute the plans and their intersection : 0, 1 or 2 $\sqrt{}$, unavoidable

Else

Parameterize through a $(2,2)$ quadric : one $\sqrt{\delta}$

$\Longrightarrow$ Equation in the parameters factors (GCD computation)

in 2 univariate factors of degree 2.

Each linear factor parameterizes one line with at most 2 $\sqrt{}$

(all together 3 $\sqrt{}$)

If the roots of $\det(\lambda\mathcal{P}+\mu Q)$ generates the same field as $\sqrt{\delta}$

Plug the parametrization of the quadric in each plan of a pair :

2 $\sqrt{}$ all together Always optimal