Hypersequents and Countermodels in Gödel-Dummett Logics

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Gödel-Dummett logics

- Intermediate logic: $\boxed{\mathrm{IL} \subset \mathrm{LC} \subset \cdots \subset \mathrm{LC}_n \subset \cdots \subset \mathrm{LC}_1 = \mathrm{CL}}$
- Syntactic characterization: $\mathrm{LC} = \mathrm{IL} + (X \supset Y) \vee (Y \supset X)$
- Semantic models:
  - Linear Kripke models or the lattice $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$
  - For finitary LC_n, $\overline{[0,n)} = [0,\ldots,n[\cup\{\infty\}$
  - Lattice structure: min, max, ...
- Complexity: LC and LC_n are NP-complete

Deciding LC with proof-search

- There exist various calculi dedicated to proof-search in LC
  - Sequent calculi (Dyckoff, Larchey)
  - Hypersequent calculi (Avron, Metcalfe et al.)
  - Sequent of relations calculi (Baaz et al.)
  - Relational hypersequent calculi (Fermüller)
- **Proof-search and countermodel generation combined**
  - Strongly invertible rules to reduce (hyper)sequents
  - Semantic computation to decide irreducible (hyper)sequents

Deciding LC with proof-search (2)

- A recent contribution propose a similar approach (Larchey)
  - strongly invertible proof rules for sequents
  - Decide irreducible sequents with bi-colored graphs
- **Strong invertibility** of logical rules
  - Preserves countermodels from premises to conclusion
  - No backtracking in proof-search
  - Countermodel generation

Overview

- Hypersequents (single-conclusion)
- Basic Hypersequents
- Bi-colored semantic graphs:
  - Basic hypersequents
  - R-cycles
  - $(n+1)$-alternating chains
  - Height and countermodel
- Decision procedure:
  - LC and LC_n
  - The rules of GLC^* system
  - Countermodel generation

Overview (2)

- A new system for the finitary case LC_n
  - Extension of the GLG^* system (LC)
  - n-generalized axioms
- A new tableau system for LC_n
  - Extension of Avron's tableau system for LC
- Bi-colored graphs and hypersequents

Plan

1. Hypersequents
2. Procedure for basic hypersequents
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Hypersequent (1)

- Multiset of sequents
- $\mathcal{H} = \Gamma_1 \vdash C_1 \mid \ldots \mid \Gamma_m \vdash C_m$
- An interpretation: $[\![\cdot]\!] : \mathrm{Var} \to \overline{[0, n)}$
- $\lfloor\!\lfloor A_1, \ldots, A_k \rfloor\!\rfloor = min([\![A_1]\!] \ldots, [\![A_k]\!]$
- $\mathcal{H}$ is valid in LC_n iff for all interpretation $[\![\cdot]\!]$, $\exists i$, $\lfloor\!\lfloor \Gamma_i \rfloor\!\rfloor \leq [\![C_i]\!]$

$\leadsto$ $[\![\cdot]\!]$ is a countermodel of $\mathcal{H}$ in LC_n iff $\forall i$, $\lfloor\!\lfloor \Gamma_i \rfloor\!\rfloor > [\![C_i]\!]$

Hypersequent (2)

- Basic hypersequent
  - Introduced by Avron
  - Particular calss of hypersequents
  - The components
    - $\Gamma \vdash p$ where p and any element of Γ are atoms
    - $p \rightarrow q \vdash p$ where p and q are atoms and $p \neq q$, $p \neq \bot$

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  - Nodes:
    - The variables
    - A node denoted $\diamondsuit$
    - $\bot$ (it contains $\bot$)
  - Arrows:
    - $p \rightarrow q \vdash p \rightsquigarrow \{p \rightarrow q,\ p \Rightarrow \diamondsuit\}$
    - $q_1, \ldots, q_m \vdash p \rightsquigarrow \{p \Rightarrow q_1, \ldots, p \Rightarrow q_m\}$
    - it contains $\bot \rightsquigarrow \{\bot \rightarrow p$, for any $p \in \mathrm{Var}\}$

Deciding/refuting basic hypersequents in LC

- An example

$$A \to B \vdash A \mid A \vdash B$$

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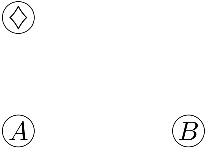

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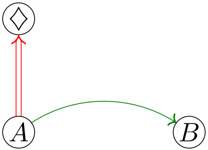

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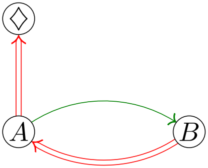

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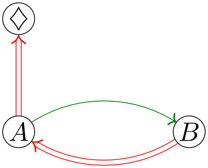

- **R-cycle**: cycle with at least one red arrow $(x(\rightarrow + \Rightarrow)^{\star}\Rightarrow x)$.

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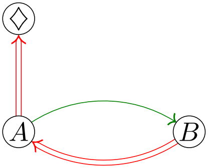

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Theorem

A basic hypersequent $\mathcal{H}$ has a countermodel in LC iff its bi-colored graph does not contain a r-cycle.

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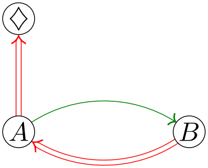

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$\rightsquigarrow A \rightarrow B \Rightarrow A$

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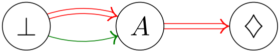

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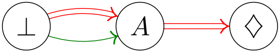


- 2-alternating chain: $(\rightarrow^{\star}\Rightarrow)^2$ $(\bot \Rightarrow A \Rightarrow \Diamond)$

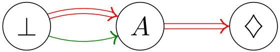
$\leadsto$ $\vdash A \mid A \vdash \bot$ is valid in LC_1

Countermodel generation

- The graphe does not contain a r-cycle.
  - Draw the graph **by levels** (linear time) :
    - Red arrows $\Rightarrow$ go up (strictly)
    - Blue arrows $\rightarrow$ never go down
  - The counter model is given by the height
  - A basic hypersequent has a countermodel in LC_n iff its graph has **at most** $n+1$ **levels** (height $\leqslant n$)

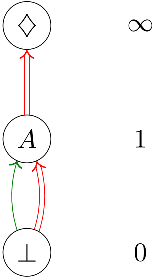
Example

- $\vdash A \mid A \vdash \bot$



- There is no chain of the form $(\rightarrow^{\star}\Rightarrow)^n$ for $n > 2$

- Draw the graph **by levels**:



$\rightsquigarrow$ $[\![A]\!] = 1$ is a coutermodel of $\vdash A \mid A \vdash \bot$ in LC_n for every $n > 1$

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Some rules of GLC^*

- introduced by Avron
- The irreducible hypersequents: basic hypersequents

$$\frac{G \mid \Gamma \vdash r \mid p \rightarrow q \vdash p \qquad G \mid \Gamma, q \vdash r}{G \mid \Gamma, p \rightarrow q \vdash r}\ [\rightarrow_L] \qquad \frac{G \mid \Gamma, A \vdash B}{G \mid \Gamma \vdash A \rightarrow B}\ [\rightarrow_R]$$

$$\frac{G \mid A \vdash B \mid \Gamma, B \rightarrow C \vdash D \qquad G \mid \Gamma, C \vdash D}{G \mid \Gamma, (A \rightarrow B) \rightarrow C \vdash D}\ [(\rightarrow)\rightarrow_L]$$

$$\frac{G \mid \Gamma, A \rightarrow C \vdash D \qquad G \mid \Gamma, B \rightarrow C \vdash D}{G \mid \Gamma, A \rightarrow (B \rightarrow C) \vdash D}\ [\rightarrow(\rightarrow)_L]$$

Decision procedure for hypersequents

- For every G we can effectively find a set $\mathcal{B}$ of basic hypersequents by using the rules of GLC^*, so that G is valid iff H is valid for every $H \in \mathcal{B}$
- The use of the bi-colored graphs to decide the elements of $\mathcal{B}$
- The rules of GLC^* are strongly invertible

$\rightsquigarrow$ for any $H \in \mathcal{B}$, if $[\![\cdot]\!] : \mathrm{Var} \longrightarrow \overline{[0, n)}$ is countermodel of H then $[\![\cdot]\!]$ is countermodel of G

- builds countermodel

Example (1/3)

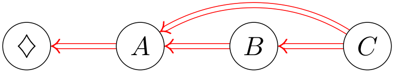
- $\mathcal{H} = \vdash A \vee (A \rightarrow B) \vee ((A \wedge B) \rightarrow C)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash A \mid A \vdash B \mid A, B \vdash C}{\vdash A \mid A \vdash B \mid A \wedge B \vdash C}[\wedge_L]}{\vdash A \mid A \vdash B \mid \vdash (A \wedge B) \rightarrow C}[\rightarrow_L]}{\vdash A \mid \vdash A \rightarrow B \mid \vdash (A \wedge B) \rightarrow C}[\rightarrow_L]}{\vdash A \mid \vdash (A \rightarrow B) \vee ((A \wedge B) \rightarrow C)}[\vee_R]}{\vdash A \vee (A \rightarrow B) \vee ((A \wedge B) \rightarrow C)}[\vee_R]$$

- $\mathcal{B} = \{\vdash A \mid A \vdash B \mid A, B \vdash C\}$

Example (2/3)

- The bi-colored graph of $\mathcal{H}_{\mathcal{B}} = \vdash A \mid A \vdash B \mid A, B \vdash C$:



- A 3-alternating chain $C \Rightarrow B \Rightarrow A \Rightarrow \Diamond$

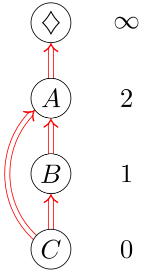
$\rightsquigarrow$ $\mathcal{H}_{\mathcal{B}}$ is valid in $\mathrm{LC}_2 \Rightarrow \mathcal{H}$ is valid in LC_2

Example (3/3)

- There is no 4-alternating chain

⇝ $\mathcal{H}_\mathcal{B}$ has a countermodel in $(\mathsf{LC}_n)_{n>2} \Rightarrow \mathcal{H}$ has a countermodel in $(\mathsf{LC}_n)_{n>2}$

- Draw the graph by levels:



⇝ $[\![\cdot]\!] : \mathsf{Var} \to \{0, 1, \infty\}$ s.t. $[\![A]\!] = 2$, $[\![B]\!] = 1$ and $[\![C]\!] = 0$ is a coutermodel of $\mathcal{H}$ in $(\mathsf{LC}_n)_{n>2}$

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The GLC^* sytem

- **Generalized axiom:**
  - A basic hypersequent
  - $p_1 \prec p_2 \mid p_2 \prec p_3 \mid \ldots \mid p_{n-1} \prec p_n \mid p_n \vdash p_1$ where $p_i \prec p_{i+1}$ is either $p_i \vdash p_{i+1}$ or $(p_{i+1} \rightarrow p_i) \vdash p_{i+1}$
  - $(p_1 \rightarrow \bot) \vdash p_1 \mid (p_2 \rightarrow p_1) \vdash p_2 \mid \ldots \mid (p_{n-1} \rightarrow p_{n-2}) \vdash p_{n-1} \mid p_{n-1} \vdash p_n$
  - Exemples: $\bot \vdash \bot$, $p \vdash p$, $p \vdash q \mid q \vdash p$
- **Axioms:** Every basic hypersequent wich can be derived from some generalized axiom using (internal and external) weakenings

$$\frac{G}{G \mid \Gamma \vdash A}\ [ew] \qquad \frac{G \mid \Gamma \vdash C}{G \mid \Gamma, A \vdash C}\ [iw]$$

- **Rules:**

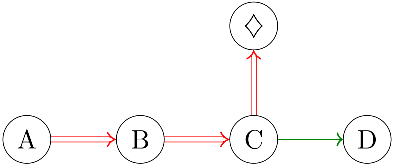
n-generalized axiom

- A basic hypersequent
- generalized axiom
- $p^1_{m_1} \vdash p^1_{m_1-1} \mid (p^2_1 \rightarrow p^2_2) \vdash p^2_1 \mid (p^2_2 \rightarrow p^2_3) \vdash p^2_2 \mid \ldots \mid$
  $(p^2_{m_2-2} \rightarrow p^2_{m_2-1}) \vdash p^2_{m_2-2} \mid p^2_{m_2} \vdash p^2_{m_2-1} \mid (p^3_1 \rightarrow p^3_2) \vdash p^3_1 \mid$
  $(p^3_2 \rightarrow p^3_3) \vdash p^3_2 \mid \ldots \mid (p^3_{m_3-2} \rightarrow p^3_{m_3-1}) \vdash p^3_{m_3-2} \mid p^3_{m_3} \vdash p^3_{m_3-1} \mid$
  $\ldots$
  $(p^n_1 \rightarrow p^n_2) \vdash p^n_1 \mid (p^n_2 \rightarrow p^n_3) \vdash p^n_2 \mid \ldots \mid$
  $(p^n_{m_n-2} \rightarrow p^n_{m_n-1}) \vdash p^n_{m_n-2} \mid p^n_{m_n} \vdash p^n_{m_n-1} \mid p^n_{m_n} \vdash' p_f$

n-generalized axiom

- The simplest basic hypersequents the bi-colored graphs of wich contain a $n+1$-alternating chain
- An example (2-generalized axiom):

$B \vdash A \mid C \vdash B \mid C \rightarrow D \vdash C$



The GLC_n^* system

- Finitary versions of Gödel-Dummett logic $(\mathrm{LC}_n)_{n>0}$
- **Axioms:** Every basic hypersequent wich can be derived from some n-generalized axiom using (internal and external) weakenings

$\rightsquigarrow$ All the basic hypersequents valid in LC_n

- **Rules:** the rules of GLC^*

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A tableau system for finitary versions

- $\mathcal{F}$ is valid in GLC_n^* iff $\vdash \mathcal{F}$ has a proof in GLC_n^*
- Obtained from Avron's tableau system for LC based on GLC^*
- We only have to chage the definition of closed branchs by using the axioms of the GLC_n^* system

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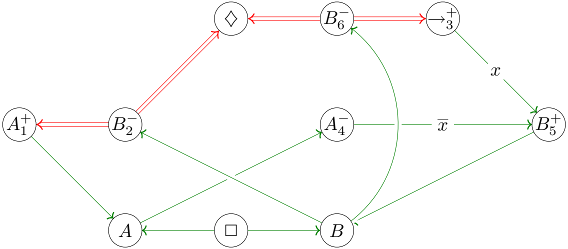
Bi-colored graphs and hypersequents

- Bi-colored graphs associated to hypersequents
  - Transformation of hypersequents into flat sequents (indexing process)
  - Transformation of flat sequents into conditional bi-colored graphs (arrows indexed with a boolean selector)
  - Instance graphs obtained by setting selectors ($x = 0$ or 1)

- **Results**: characterization of provability
  - $\mathcal{H}$ is provable in LC iff every instantce graph has a r-cycle
  - $\mathcal{H}$ is provable in LC iff every instantce graph has a $(n+1)$-alternating chain
  - if an instance has no $(n+1)$-alternating chain (resp. r-cycle), its height is a countermodel in LC_n (resp. LC)

Example

- $\mathcal{H} = A \vdash B \mid A \rightarrow B \vdash B$

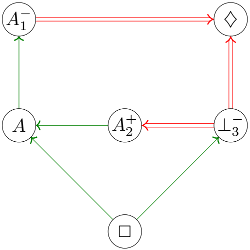


- There are two instances ($x = 0$ and $x = 1$)
  - $x = 0$: $B_2^- \Rightarrow A_1^+ \rightarrow A \rightarrow A_4^- \rightarrow B_5^+ \rightarrow B \rightarrow B_2^-$
  - $x = 1$: $B_6^- \Rightarrow \rightarrow_3^+ \rightarrow B_5^+ \rightarrow B \rightarrow B_6^-$

$\rightsquigarrow$ $\mathcal{H}$ is valid in LC

An example with countermodel generation (1/2)

- $\mathcal{H} = \vdash A \mid A \vdash \bot$

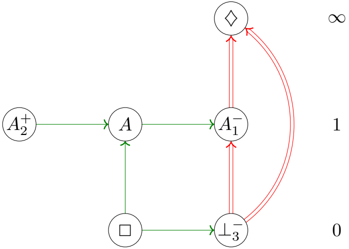


- There is no r-cycle
- There is a 2-alternating chain $(\bot_3^- \Rightarrow A_2^+ \Rightarrow \Diamond)$

$\rightsquigarrow$ $\mathcal{H}$ is valid in LC_1

An example with countermodel generation (2/2)

- Draw the graph by levels:



- $[\![\cdot]\!]$: Var $\to \{0, 1, \infty\}$ s. t. $[\![A]\!] = 1$ is a countermodel of $\mathcal{H}$ in $(\mathsf{LC}_n)_{n>1}$

Conclusion

- New characterizations of validity in LC and LC_n
- Countermodel generation