On the extension of computable real functions

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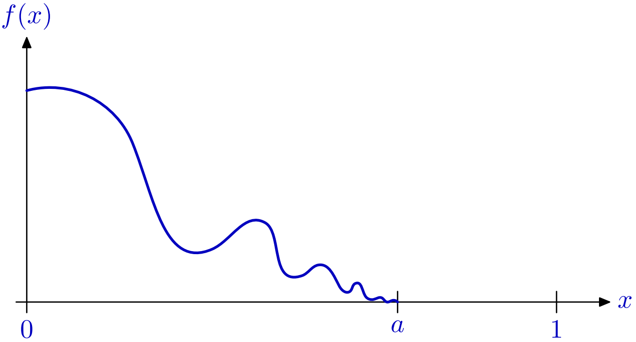
The problem

Let $f : [0, a) \to \mathbb{R}$ be computable and converge to 0 at $a \in (0, 1)$.



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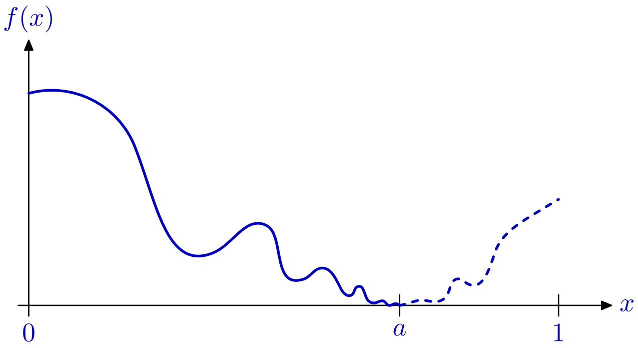


When can f be extended to a computable function over $[0, 1]$?

The problem

In real analysis

The following are equivalent:

- $f : [0, a) \to \mathbb{R}$ has a continuous extension,
- f converges at a,
- f is uniformly continuous.

In computable analysis

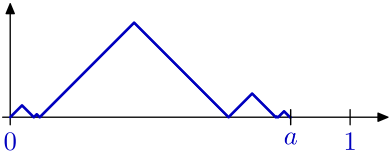
Assuming a is **computable**, the following are equivalent:

- $f : [0, a) \to \mathbb{R}$ has a **computable** extension,
- f converges **effectively** at a,
- f is **effectively** uniformly continuous.

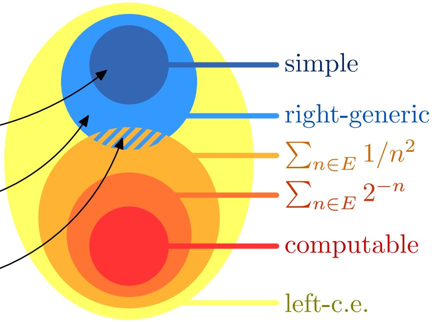
Questions

What if a is not computable? For which a's does the equivalence hold?

Our results



- Complete answer for the class of **sawtooth functions**.
- **Sufficient** and **necessary** conditions (S) and (N).
- Characterization of the a's for which (S) is also **necessary**.
- Characterization of the a's for which (N) is also **sufficient**.
- **Separation** of these two classes.



$E \subseteq \mathbb{N}$ is any recursively enumerable set

A few definitions

We study the following cases:

- When a is **computable**, easy.
- When a is **left-c.e.**, more interesting.

Definition

- $a \in \mathbb{R}$ is ***computable*** if there is a computable rational sequence $(a_i)_{i\in\mathbb{N}}$ such that:

$$\forall i, |a_i - a| \leq 2^{-i}.$$

- $a \in \mathbb{R}$ is ***left-c.e.*** if there is a computable rational sequence $(a_i)_{i\in\mathbb{N}}$ such that:

$$a_i \nearrow a.$$

- $f : [0, a) \to \mathbb{R}$ is ***computable*** if $f(x)$ can be computed with arbitrary precision, given $x \in [0, a)$ with arbitrary precision.

First observations

Let

- $a \in [0, 1]$ be left-c.e., non-computable,
- $f : [0, a) \to \mathbb{R}$ be computable.

If f has a computable extension g on $[0, 1]$, what can g look like?

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Example

If f is 1-Lipschitz then g is 1-Lipschitz on $[0, a + \epsilon]$ for some $\epsilon > 0$.

First observations

Two computable extensions g, h must agree on $[0, a+\epsilon]$ for some $\epsilon > 0$:

Proof.

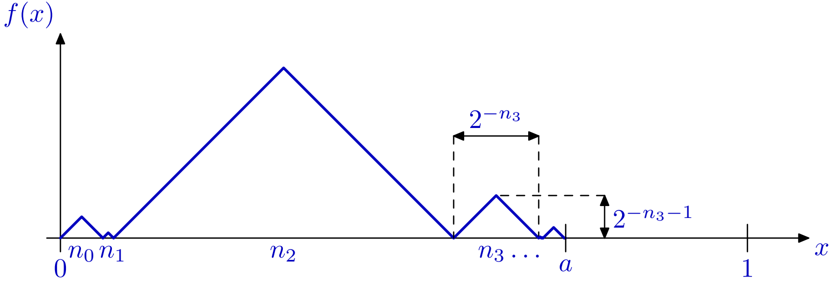
If g, h are computable extensions of f then

$$b := \inf\{x \in [0,1] : g(x) \neq h(x)\}$$

is right-c.e. and $b \geq a$, so $b > a$. □

Sawtooth functions

- Take a recursively enumerable set $E \subseteq \mathbb{N}$,
- Take a computable enumeration $n_0, n_1, n_2, \ldots$ of E,
- Define $a = \sum_{n\in E} 2^{-n}$ and the sawtooth function f:



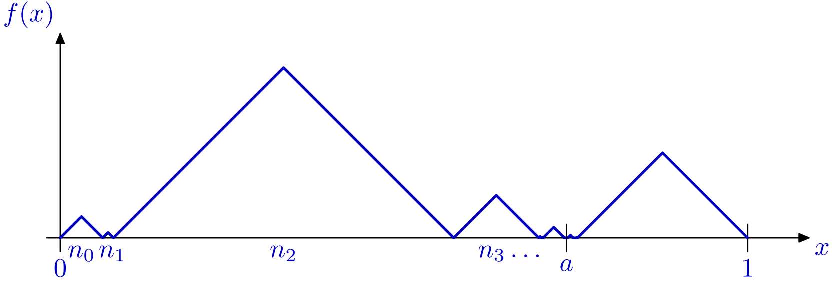
When does f have a computable extension?

Sawtooth functions

Theorem

The function associated with $E = \{n_0, n_1, \dots\}$ has a computable extension $\Leftrightarrow$ there exists a computable linear ordering $\preceq$ over $\mathbb{N}$ s.t.:

- $n_0 \prec n_1 \prec n_2 \prec \dots$
- E *is an initial segment:* $n \prec p$ *for* $n \in E, p \notin E$.



Key ingredient: being sawtooth is a **recursively compact** property.

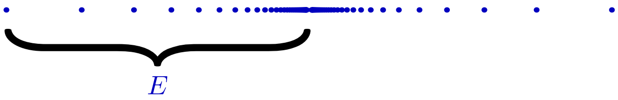
Sawtooth functions

Negative case

Let E be the halting set. It is not an initial segment of a computable linear ordering.

Positive case

There exists a computable linear ordering of order type $\omega + \omega^*$ whose left part is a recursively enumerable, non-computable set E.



Sufficient condition

Definition

f ***converges effectively*** to 0 (at a) if

given $\epsilon > 0$ one can compute $q < a$ such that $|f| \leq \epsilon$ on $[q, a)$.

Examples

- If f **decreases** to 0 then f converges effectively to 0.
- Let $E \subseteq \mathbb{N}$ be recursively enumerable. The sawtooth function f_E converges effectively to 0 iff E is computable.

Proposition

f *converges effectively to* 0 *iff its null extension is computable.*

Necessary condition

Definition

A function f is ***effectively uniformly continuous*** if given $\epsilon > 0$ one can compute $\delta > 0$ such that

$$|x - y| \leq \delta \implies |f(x) - f(y)| \leq \epsilon.$$

Example

Every Lipschitz function (hence every sawtooth function) is effectively uniformly continuous: take $\delta := \epsilon / L$.

Proposition

If f has a computable extension then f is eff. unif. cont.

Implications

Condition (S)

f converges effectively to 0

Condition (Ext)

f has a computable extension on $[0,1]$.

Condition (N)

f is effectively uniformly continuous.

When a is computable

$$(S) \iff (Ext) \iff (N)$$

When a is left-c.e.

$$(S) \implies (Ext) \implies (N)$$

The implications are strict: consider the two sawtooth functions defined earlier ($E =$ halting set/initial segment of linear ordering ...). For which a's are they strict, exactly?

Dependence on a

Condition (S)

f converges effectively to 0.

Condition (Ext)

f has a computable extension on $[0, 1]$.

Condition (N)

f is effectively uniformly continuous.

$$\text{(S)} \implies \text{(Ext)} \implies \text{(N)}$$

Theorem

*One has (S) $\iff$ (Ext) exactly when a is **right-generic or computable**.*

Theorem

*One has (Ext) $\iff$ (N) exactly when a is **simple**.*

Right-generic reals

Definition

A real is **right-generic** if it is not contained in any "computable small set".

Nonexamples

- Computable x.

- $x_E := \sum_{n \in E} 2^{-n}$, where $E \subseteq \mathbb{N}$ is recursively enumerable.

Figure: The set of reals whose bits at positions in E are all 1.

Simple reals

A **presentation** of $a \in [0,1]$ is a prefix-free recursively enumerable set $A \subseteq \{0,1\}^*$ such that

$$a = \sum_{u \in A} 2^{-|u|}.$$

Definition ([Downey, LaForte 2002])

A left-c.e. real a is **simple** if every presentation of a is computable.

Nonexamples

- $x_E := \sum_{n \in E} 2^{-n}$, where $E \subseteq \mathbb{N}$ is recursively enumerable, not computable.
- $\Omega_U := \sum_{w \in \mathrm{dom}(U)} 2^{-|w|}$, where U is a universal prefix-free Turing machine.

Simple vs right-generic

Proposition

Simple $\Longrightarrow$ *right-generic.*

What about the other direction?

Simple vs right-generic

Proposition

Simple $\Longrightarrow$ right-generic.

What about the other direction?

Theorem

Simple $\not\Longleftarrow$ right-generic.

Proof idea.

Let $E \subseteq \mathbb{N}$ be a non-computable c.e. set.

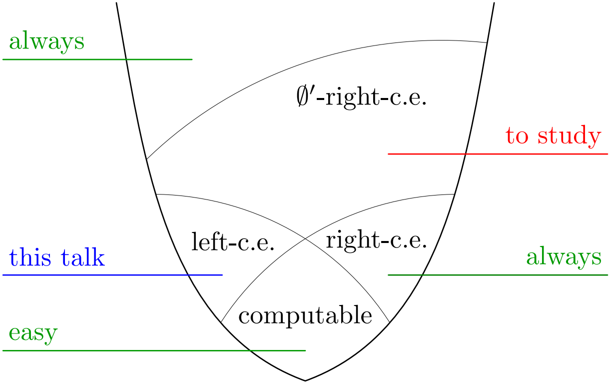
- $x_E := \sum_{n \in E} 2^{-n}$ is not simple, and not even right-generic.
- $y_E := \sum_{n \in E} \frac{1}{n^2}$ is not simple. **It can be right-generic.** □

Other a's

We have studied the computable extension problem when a is:

- Computable,
- Left-c.e.

What about other cases?



To conclude

Sum up

- Rich problem,
- Unexpected relationships with many concepts from computability theory,
- Characterizations of classes of reals via computable analysis.

Many questions left

- When can $f : [0, a) \to \mathbb{R}$ be extended to $[0, a]$?
- When can $f : [0, a] \to \mathbb{R}$ be extended to $[0, 1]$?
- What if $\lim_{x \to a^-} f(x) \neq 0$?
- What if f is non-decreasing?
- What happens when a is $\emptyset'$-right-c.e.?
- ...

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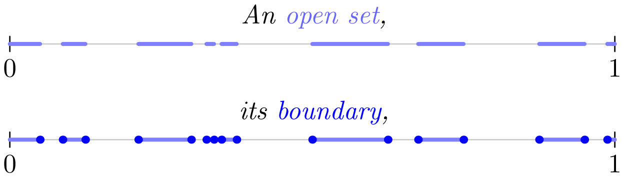
Thanks!

Generic reals

- A real $x \in [0, 1]$ is 1**-generic** if it does not belong to the boundary of any effective open set [Jockusch 1977].



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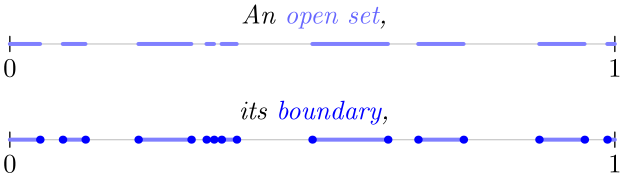

(Non-)examples

- No **left-c.e.** real x is 1-generic, as $[0,x)$ is effectively open.

- No **right-c.e.** real x is 1-generic, as $(x,1]$ is effectively open.

Generic reals

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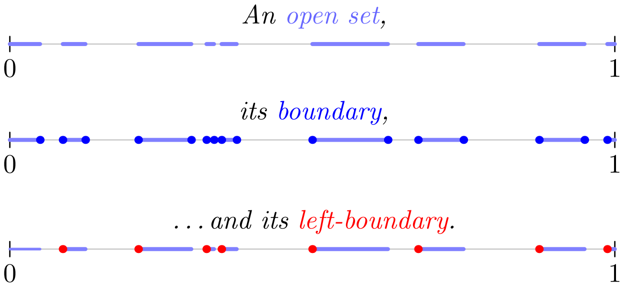

- A real $x \in [0,1]$ is **right-generic** if it does not belong to the left-boundary of any effective open set [H. 2014].

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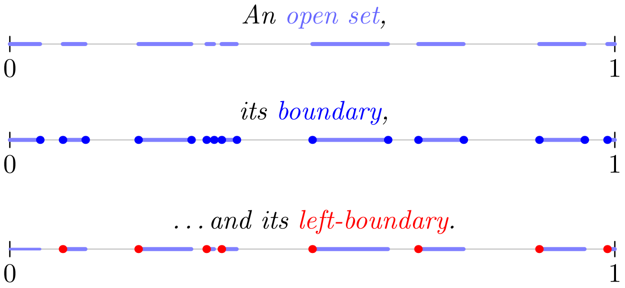

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Theorem ([H. 2014])

Right-generic left-c.e. reals exist.

Generic reals

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$$x_E := \sum_{n \in E} 2^{-n}$$

is not right-generic.

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$E = \{ \qquad \}$

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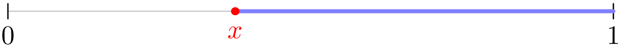

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$E = \{2, 5, 7, 9 \dots\}$