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Abstract: In this paper, we show that, amongst n uniformly distributed unit balls in $\mathbb{R}^3$, the expected number of maximal non-occluded line segments tangent to four balls is linear, considerably improving the previously known upper bound. Using our techniques we show a linear bound on the expected size of the visibility complex, a data structure encoding the visibility information of a scene, providing evidence that the storage requirement for this data structure is not necessarily prohibitive.

Our results generalize in various directions. We show that the linear bound on the expected number of maximal non-occluded line segments that are not too close to the boundary of the scene and tangent to four unit balls extends to balls of various but bounded radii, to polyhedra of bounded aspect ratio, and even to non-fat 3D objects such as polygons of bounded aspect ratio. We also prove that our results extend to other distributions such as the Poisson distribution. Finally, we indicate how our probabilistic analysis provides new insight on the expected size of other global visibility data structures, notably the aspect graph.

Key-words: computational geometry, 3D visibility, visibility complex, visual events, probabilistic analysis, expected complexity

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Le nombre moyen d'évènements visuels 3D est linéaire

Résumé : Nous montrons que, pour une scène de $\mathbb{R}^3$ composée de n boules de rayon 1 uniformément distribuées, le nombre moyen de segments de droites non occultés, maximaux et tangents à quatre boules est linéaire. Nous montrons que la borne linéaire s'applique également à la taille moyenne du complexe de visibilité, une structure de données encodant l'information de visibilité de la scène, donnant à penser que l'espace mémoire nécessaire au stockage de cette structure de données n'est pas nécessairement prohibitif.

Nous généralisons ces résultats dans différentes directions. Nous montrons que la borne linéaire sur le nombre moyen de segments de droites non occultés, maximaux et tangents à quatre boules unités suffisamment éloignées du bord de la scène s'applique également à des scènes composées de boules de rayon différent mais borné, de polyèdres de proportion bornée et aussi à des objets non épais comme des polygones de proportion bornée. Nous prouvons également que nos résultats se généralisent à d'autres distributions, notamment la distribution de Poisson. Enfin, nous montrons comment nos résultats ouvrent de nouvelles perspectives concernant l'étude de la taille moyenne d'autres structures de visibilité globale comme le graphe d'aspects.

Mots-clés : géométrie algorithmique, visibilité 3D, complexe de visibilité, évènements visuels, analyse probabiliste, complexité en moyenne

1 Introduction

Visibility computations are central in computer graphics applications. Computing the limits of the umbra and penumbra cast by an area light source, identifying the set of blockers between any two polygons and determining the view from a given point are examples of visibility queries that are essential for the realistic rendering of 3D scenes. In global illumination algorithms, where the flow of light in a scene is simulated according to the laws of geometrical optics, visibility computations are excessively costly. In fact, more than half of the overall computation time can routinely be spent on visibility queries in radiosity simulations [12].

One approach to speeding up rendering is to store global visibility information in a data structure which can then be efficiently queried. The visibility complex, a partition of the set of maximal free line segments, has been proposed as a unified data structure encoding the visibility information of a scene [21] and has been used for rendering purposes [10]. Other related data structures include Pellegrini's ray-shooting structure [18], the aspect graph [20] and the visual hull [13]; see [7] for a recent survey.

One problem with these types of data structures which may prevent their application in practice is their potentially enormous size; the size of the visibility complex of a set of n triangles in $\mathbb{R}^3$ is $\Theta(n^4)$ in the worst case [10], which is prohibitive even for scenes of relatively modest size. Worst-case examples are somewhat artificial and indeed Durand, Drettakis and Puech [8] provide empirical evidence indicating that these worst-case upper bounds are largely pessimistic in practical situations; they observe a quadratic growth rate, albeit for rather small scenes. In 2D, while the worst-case complexity of the visibility complex is quadratic, experimental results strongly suggest that the size of the visibility complex of a scene consisting of scattered triangles is linear [3].

Our goal is to provide theoretical evidence to support these observations. To this end we investigate the *expected size* of the visibility complex, or equivalently, the expected number of visibility events, occurring in scenes in $\mathbb{R}^3$. A visibility event is a combinatorial change in the view of a moving observer; such an event occurs when the viewing direction becomes tangent to some objects. For sets of convex objects in general position in $\mathbb{R}^3$, the viewing direction can be tangent to at most four objects. Visibility events thus correspond to maximal non-occluded line segments tangent to at most four objects; combinatorially different visibility events correspond to the faces of the visibility complex.

In this paper we prove that the expected number of maximal non-occluded line segments tangent to four balls, amongst n uniformly distributed unit balls in $\mathbb{R}^3$, is linear, considerably improving the result of Durand et al. [10] who proved a bound of $O(n^{8/3})$ for the same model. The intuition behind our proof is that, given a line segment tangent to four balls, the probability that this segment is not occluded by any other ball is the probability that a cylinder-like volume of radius 1 about the segment is free from the centers of the other balls. This probability decays roughly exponentially fast with the length of the segment, yielding the result. Using our techniques we then show a linear bound on the expected size of the visibility complex of n uniformly distributed unit balls in $\mathbb{R}^3$. A simple computation then provides us with the same result for the Poisson distribution.

	Worst-case	Expected
possibly occluded lines amongst unit balls	$\Theta(n^4)$	$O(n^{\frac{8}{3}})$ [10]
free lines amongst unit balls	$\Omega(n^2)$ [1, $\star$], $O(n^{3+\epsilon})$ [1]	$\Theta(n)$ [$\star$]
free lines amongst disjoint homothetic polytopes	$\Omega(n^3)$ [2]	?
free segments amongst unit balls	$\Omega(n^2)$ [1, $\star$], $O(n^4)$	$\Theta(n)$[$\star$]
free segments amongst arbitrary sized balls	$\Omega(n^3)$ [5], $O(n^4)$	?
visibility complex of unit balls	$\Omega(n^2)$ [$\star$], $O(n^4)$	$\Theta(n)$ [$\star$]

Table 1. Known bounds on the complexity of the set of lines, free lines or maximal free line segments tangent to 4 amongst n objects. The expected complexities are calculated for the uniform distribution. The results referenced by $\star$ are established in this paper.

Our results generalize in the following ways. We show that the linear bound on the expected number of visibility events occurring not too close to the boundary of the scene also applies to balls of various but bounded radii, to polyhedral objects enclosed between two concentric balls of fixed radius, and even to non-fat objects such as polygons, enclosed between two concentric circles of fixed radius, whose centers and normals are uniformly distributed.

Of course objects in graphics scenes are seldom distributed uniformly or according to a Poisson point process. We chose this model because it allows tractable proofs of theoretical results. This is important in a context where there are few rigorous results either theoretical or experimental. The same model has also been used to study the average complexity of ray shooting [23] and occlusion culling for 2D urban scenes [16]. No other model has been widely accepted by the graphics community and, in fact, generating random scenes usable for testing algorithms is a major problem.

Previous results on this topic include those that bound the number of lines and the number of free (i.e., non-occluded) lines amongst different sets of objects. They are summarized in Table 1. Agarwal, Aronov and Sharir [1] showed an upper bound of $O(n^{3+\epsilon})$ on the complexity of the space of line transversals of n balls by studying the lower envelope of a set of functions. A study of the upper envelope of the same set of functions yields the same upper bound on the number of free lines tangent to four balls [5]. Agarwal et al. [1] also showed a lower bound on the complexity of the space of line transversals of n balls of $\Omega(n^3)$ for arbitrarily sized balls and $\Omega(n^2)$ for unit sized balls. The construction also shows the same bounds for the number of lines tangent to four balls. De Berg, Everett and Guibas [2] showed a $\Omega(n^3)$ lower bound on the number of free lines (and thus free segments) tangent to four amongst n disjoint homothetic convex polyhedra. Recently, Devillers and Ramos [5] presented a simple $\Omega(n^3)$ lower bound on the number of free segments tangent to 4 amongst n arbitrarily sized balls, which also holds for non-intersecting balls. We also present a simple $\Omega(n^2)$ lower bound on the number of free segments tangent to 4 amongst n unit balls.

In the next section we carefully define the problem and state our main results. In Section 3 and Section 4 we prove the expected upper and lower linear bounds on the number of free segments tangent to four balls. In Section 5 we extend this result to the visibility complex. We present in Section 6 a $\Omega(n^2)$ worst-case lower bound. In Section 7 we discuss extensions of our results to some other models. We conclude in Section 8.

2 Our model and results

We first describe our objects and their distribution. Let $n \in \mathbb{N}$ and μ be a positive constant. A sample scene consists of n unit radius balls $B_1, \ldots, B_n$ whose centers $p_1, \ldots, p_n$ are independently chosen from the uniform distribution over a universal ball $\mathcal{U}$ of radius R centered at O. Since we distribute the centers p_i over $\mathcal{U}$, the balls B_i may intersect each other and are contained in the ball, denoted $\mathcal{U}^+$, whose radius is $R+1$ and whose center is that of $\mathcal{U}$.

We define the radius R of the universal ball $\mathcal{U}$ to be a function of n satisfying

$$R^3 = n/\mu. \tag{1}$$

The constant μ reflects the density of the balls in the sense that the expected number of centers lying in any given solid of volume V in the universe is $\frac{3}{4\pi}\mu V$. (The model is interesting only if n is asymptotically proportional to R^3. Indeed, if $\frac{n}{R^3}$ tends to infinity when n tends to infinity, then the universe gets entirely filled up with balls and visibility events only occur in $\mathcal{U}^+ \setminus \mathcal{U}$. Conversely, if $\frac{n}{R^3}$ tends to zero when n tends to infinity, then the balls get scattered so far apart that the probability that any four (or three) balls have a common tangent goes to zero.)

We now define the *visibility complex* of a set of objects [21]. A *free* or *non-occluded* segment is a line segment that does not intersect the interior of any object. A free segment is maximal if it is not properly contained in another one. Thus, the endpoints of a maximal free segment are either on an object or at infinity. We say that two maximal free segments are similar if their endpoints lie on the same objects (possibly at infinity). The visibility complex of a collection of objects is roughly defined as the partition of the space of maximal free segments into connected components of similar segments[1]. Its faces have dimension between 0 and 4; when the objects are in adequate general position, a k-dimensional face corresponds to a connected set of similar maximal non-occluded line segments tangent to $4-k$ objects.

In order to bound the total number of faces of the visibility complex, we first bound the number of 0-faces. To do this, we count the $T4$-*segments*, which are the free segments tangent to 4 balls with endpoints on two of those balls. Since there is a one-to-one correspondence between 0-faces and $T4$-segments when the objects are in adequate general position, this yields a bound on the expected number of vertices of the visibility complex. Note that since the balls are contained in $\mathcal{U}^+$, the $T4$-segments are also contained in $\mathcal{U}^+$.

Our main result is the following.

[1]Formally, we consider the space of free segments quotiented by the equivalence relation that is the transitive and reflexive closure of the inclusion. In other words, two free segments are identified if they are both contained in the same maximal free segment. This allows the cells of the partition to be connected.

Theorem 1 *The expected number of $T4$-segments amongst n uniformly distributed unit balls is $\Theta(n)$.*

We extend this result to the higher dimensional faces of the complex.

Theorem 2 *The expected size of the visibility complex of n uniformly distributed unit balls is $\Theta(n)$.*

We also present an $\Omega(n^2)$ worst-case lower bound on the number of $T4$-segments amongst n unit balls in $\mathbb{R}^3$ (see Proposition 27). In fact the lower bound holds for the number of k-faces of the visibility complex, for all k between 0 and 4.

3 The expected number of T4-segments is at most linear

The general idea behind the proof of the upper bound of Theorem 1 is the following. For any ordered choice of four balls, we bound from above the probability that a line is tangent to these balls in the given order and is not occluded in between its contact points with the balls. Then we sum these probabilities over all ordered quadruples of balls and all potential tangent lines to these balls.

For any two points p and q, and positive real number α, let $\mathcal{H}(p,q,\alpha)$ denote the union of all the balls of radius α centered on the line segment pq (see Figure 1). We first show that a line is tangent to four balls B_i, B_j, B_k and B_l in that order only if p_j and p_k are in $\mathcal{H}(p_i,p_l,2)$. Thus the volume of $\mathcal{H}(p_i,p_l,2)\cap\mathcal{U}$ gives an upper bound on the probability that a line tangent to the four balls, in the given order, exists.

We next show that a segment tangent to four balls B_i, B_j, B_k and B_l in that order, at points t_i, t_j, t_k and t_l, respectively, is not occluded if and only if the centers of all remaining balls are outside or on the boundary of $\mathcal{H}(t_i,t_l,1)$. The volume of $\mathcal{U}\setminus\mathcal{H}(t_i,t_l,1)$ gives an upper bound on the probability that the tangent segment is not occluded. Thus, to get an upper bound on that probability, we need a lower bound on the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$.

To bound the probability that a $T4$-segment exists, we integrate over the distance between p_i and p_l, and over the distance from p_i to the boundary of the universe $\mathcal{U}$. This integral is split into three parts covering the cases where
(i) B_i and B_l are close to one another,
(ii) at least one of B_i and B_l is entirely inside the universe,
(iii) B_i and B_l are not close to one another and both are partially outside the universe.

In each case we over-estimate the volume of $\mathcal{H}(p_i,p_l,2)\cap\mathcal{U}$ and under-estimate the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$.

3.1 Definitions

Let $\mathcal{N}$ be the set of ordered 4-tuples (i,j,k,l) chosen from $\{1,2,\ldots,n\}$ such that i,j,k,l are pairwise distinct. In our model, the probability that four centers are collinear is zero, so we

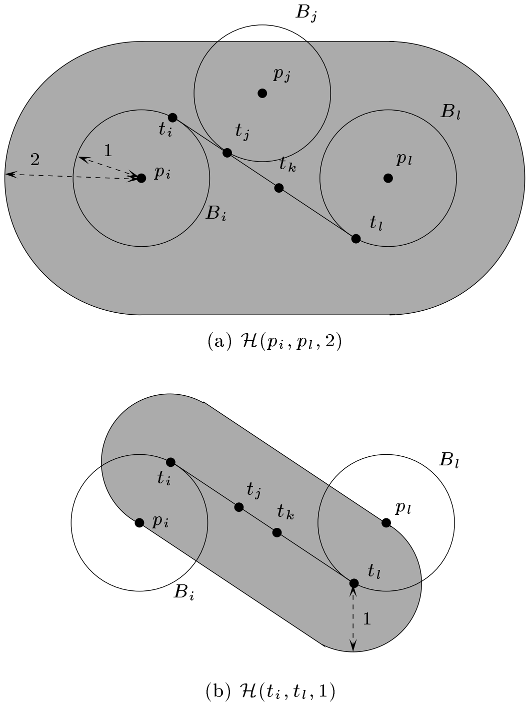


Figure 1. $\mathcal{H}(p_i, p_l, 2)$ and $\mathcal{H}(t_i, t_l, 1)$ are shown shaded.

may assume that any set of four balls admits at most 12 real common tangent lines [4, 14]. Moreover, the real common tangent lines correspond to the real solutions of a degree 12 system of equations. For any set of four balls we order arbitrarily the 12 solutions of the associated system.

Given four balls B_i, B_j, B_k and B_l, we denote by $\mathcal{L}^{\omega}_{i,j,k,l}$, for ω in $\{1, \ldots, 12\}$, the event that the ω^{th} solution of the system is real, that the corresponding real tangent line is tangent to the four balls B_i, B_j, B_k and B_l in that order, and that p_i is not closer than p_l to the boundary of $\mathcal{U}$. Whenever $\mathcal{L}^{\omega}_{i,j,k,l}$ occurs, we denote the points of tangency of that line on B_i, B_j, B_k, B_l by t_i, t_j, t_k, t_l, respectively. Let $\delta^{\omega}_{i,j,k,l}$ be the event that $\mathcal{L}^{\omega}_{i,j,k,l}$ occurs and the line segment $t_i t_l$ is not occluded. Notice that if $\delta^{\omega}_{i,j,k,l}$ occurs, the balls B_i, B_j, B_k, B_l define a $T4$-segment, and that a $T4$-segment corresponds to a unique $\delta^{\omega}_{i,j,k,l}$.

Let $\mathbf{x}_{i,l}$ be the random variable representing the distance from p_i to p_l, and $\mathbf{y}_i$ (resp. $\mathbf{y}_l$) be the random variable denoting the distance from p_i (resp. p_l) to the boundary of the universe.

In the sequel, a random point p denotes a point chosen from the uniform distribution over $\mathcal{U}$.

3.2 The Proof

There is a one-to-one correspondence between the $T4$-segments and the events $\delta^{\omega}_{i,j,k,l}$ that occur. We thus have the following straightforward lemma.

Lemma 3 *The expected number of $T4$-segments amongst n uniformly distributed unit balls is*

$$\sum_{(i,j,k,l)\in\mathcal{N}} \sum_{\omega=1}^{12} \Pr(\delta^{\omega}_{i,j,k,l}).$$

We bound the probability $\Pr(\delta^{\omega}_{i,j,k,l})$ by integrating over the distance x between p_i and p_l, and over the distance y from p_i to the boundary of the universe $\mathcal{U}$. The integral is split into three parts covering the cases where (i) the balls B_i and B_l are close to one another, (ii) p_i is at distance at least 1 from the boundary of $\mathcal{U}$, and (iii) the balls B_i and B_l are not close to one another and p_i is at distance less than 1 from the boundary of $\mathcal{U}$. Note that in the last case, if $\delta^{\omega}_{i,j,k,l}$ occurs, then both ball centers p_i and p_l are within distance 1 from the boundary of $\mathcal{U}$. Two balls are considered close to one another if their centers are closer than some sufficiently large constant; for technical reasons which are embedded in the proof of Proposition 29, we actually define *close* to mean distance at most 6.

Lemma 4 $\Pr(\delta^{\omega}_{i,j,k,l}) \leqslant I_{x\leqslant 6} + I_{y\geqslant 1} + I_{x>6,y<1}$, *where*

$$\begin{aligned}
I_{x\leqslant 6} &= \int_{x=0}^{6} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x)\cdot \Pr(x \leqslant \mathbf{x}_{i,l} < x+dx),\\
I_{y\geqslant 1} &= \int_{x=0}^{2R} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i \geqslant 1)\cdot \Pr(x \leqslant \mathbf{x}_{i,l} < x+dx \mid \mathbf{y}_i \geqslant 1),\\
I_{x>6,y<1} &= \int_{x=6}^{2R}\int_{y=0}^{1} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i)\\
&\qquad \cdot \Pr((x \leqslant \mathbf{x}_{i,l} < x+dx)\cap(\mathbf{y}_l \leqslant \mathbf{y}_i) \mid \mathbf{y}_i = y)\cdot \Pr(y \leqslant \mathbf{y}_i < y+dy).
\end{aligned}$$

Proof: By the Total Probability Theorem (see [17]),

$$\Pr(\delta^{\omega}_{i,j,k,l}) = \int_{x=0}^{2R} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x)\cdot \Pr(x \leqslant \mathbf{x}_{i,l} < x+dx).$$

The integral can be split at $x = 6$, giving $I_{x\leqslant 6}$. Then applying the Total Probability Theorem on what remains, we get

$$\int_{x=6}^{2R}\int_{y=0}^{R} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y)\cdot \Pr((x \leqslant \mathbf{x}_{i,l} < x+dx) \mid \mathbf{y}_i = y)\cdot \Pr(y \leqslant \mathbf{y}_i < y+dy) \tag{2}$$

which can be split at $y = 1$. The part corresponding to y between 1 and R is equal to

$$\begin{aligned}
&\int_{x=6}^{2R}\int_{y=1}^{R} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_i \geqslant 1) \\
&\qquad\qquad \cdot \Pr((x \leqslant \mathbf{x}_{i,l} < x + dx) \mid \mathbf{y}_i = y,\ \mathbf{y}_i \geqslant 1) \cdot \Pr(y \leqslant \mathbf{y}_i < y + dy) \\
\leqslant &\int_{x=6}^{2R}\int_{y=0}^{R} \Pr(\delta^{\omega}_{i,j,k,l} \cap (x \leqslant \mathbf{x}_{i,l} < x + dx) \mid \mathbf{y}_i = y,\ \mathbf{y}_i \geqslant 1) \cdot \Pr(y \leqslant \mathbf{y}_i < y + dy).
\end{aligned}$$

Applying the Total Probability Theorem again, we get

$$\int_{x=6}^{2R} \Pr(\delta^{\omega}_{i,j,k,l} \cap (x \leqslant \mathbf{x}_{i,l} < x + dx) \mid \mathbf{y}_i \geqslant 1)$$

which is less than $I_{y\geqslant 1}$. Consider now the part of (2) for y between 0 and 1. If $\mathbf{y}_l > \mathbf{y}_i$ then $\delta^{\omega}_{i,j,k,l}$ does not occur (by definition of $\mathcal{L}^{\omega}_{i,j,k,l}$), thus we have

$$\begin{aligned}
&\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y) \cdot \Pr((x \leqslant \mathbf{x}_{i,l} < x + dx) \mid \mathbf{y}_i = y) \\
&\quad = \Pr(\delta^{\omega}_{i,j,k,l} \cap (x \leqslant \mathbf{x}_{i,l} < x + dx) \mid \mathbf{y}_i = y) \\
&\quad = \Pr(\delta^{\omega}_{i,j,k,l} \cap (x \leqslant \mathbf{x}_{i,l} < x + dx) \cap (\mathbf{y}_l \leqslant \mathbf{y}_i) \mid \mathbf{y}_i = y) \\
&\quad = \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i) \cdot \Pr((x \leqslant \mathbf{x}_{i,l} < x + dx) \cap (\mathbf{y}_l \leqslant \mathbf{y}_i) \mid \mathbf{y}_i = y).
\end{aligned}$$

Thus, the part of (2) for y between 0 and 1 is equal to $I_{x>6,y<1}$. □

Let Ξ denote any of the following events: $(\mathbf{x}_{i,l} = x)$, $(\mathbf{x}_{i,l} = x,\ \mathbf{y}_i \geqslant 1)$, $(\mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i)$. The next three lemmas are used to bound $\Pr(\delta^{\omega}_{i,j,k,l} \mid \Xi)$ appearing in the three integrals $I_{x\leqslant 6}$, $I_{y\geqslant 1}$ and $I_{x>6,y<1}$.

Lemma 5 *If a line is tangent to four balls B_i, B_j, B_k, B_l in that order at t_i, t_j, t_k, t_l, respectively, then $p_j, p_k \in \mathcal{H}(p_i, p_l, 2)$. Also, the segment $t_i t_l$ is not occluded if and only if the interior of $\mathcal{H}(t_i, t_l, 1)$ does not contain the center of any other ball.*

Proof: Segment $t_i t_l$ is contained in $\mathcal{H}(p_i, p_l, 1)$. Since t_j and t_k belong to that segment, t_j and t_k are also in $\mathcal{H}(p_i, p_l, 1)$. Thus p_j, p_k are both in $\mathcal{H}(p_i, p_l, 2)$. See Figure 1 (a).

The segment $t_i t_l$ is occluded if and only if some ball B_γ, $\gamma \neq i, j, k, l$, properly intersects it, that is the center of B_γ lies in the interior of $\mathcal{H}(t_i, t_l, 1)$. See Figure 1 (b). □

Lemma 6 $\Pr(p \in \mathcal{H}(p_i, p_l, 2) \mid \Xi) \leqslant \dfrac{(3x+8)}{R^3}$.

Proof:

$$\Pr(p \in \mathcal{H}(p_i, p_l, 2) \mid \Xi) = \frac{\text{Volume of } \mathcal{H}(p_i, p_l, 2) \cap \mathcal{U}}{\text{Volume of } \mathcal{U}} \Big|_{\Xi} \leqslant \frac{\text{Volume of } \mathcal{H}(p_i, p_l, 2)}{\text{Volume of } \mathcal{U}} \Big|_{\Xi}.$$

When Ξ occurs, $\mathbf{x}_{i,l} = x$ and the volumes of $\mathcal{H}(p_i, p_l, 2)$ and $\mathcal{U}$ are $\frac{4\pi}{3}(3x+8)$ and $\frac{4\pi}{3}R^3$, respectively. Thus

$$\Pr(p \in \mathcal{H}(p_i, p_l, 2) \mid \Xi) \leqslant \frac{3x+8}{R^3}.$$

□

Lemma 7 $\Pr(\delta^{\omega}_{i,j,k,l} \mid \Xi) \leqslant \dfrac{(3x+8)^2}{R^6} \cdot \Pr(p \notin \mathcal{H}(t_i, t_l, 1) \mid \mathcal{L}^{\omega}_{i,j,k,l},\ \Xi)^{n-4}$.

Proof: If $\delta^{\omega}_{i,j,k,l}$ occurs, then $\mathcal{L}^{\omega}_{i,j,k,l}$ necessarily occurs, thus

$$\Pr(\delta^{\omega}_{i,j,k,l} \mid \Xi) = \Pr(\delta^{\omega}_{i,j,k,l} \cap \mathcal{L}^{\omega}_{i,j,k,l} \mid \Xi) = \Pr(\mathcal{L}^{\omega}_{i,j,k,l} \mid \Xi) \cdot \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathcal{L}^{\omega}_{i,j,k,l},\ \Xi).$$

By Lemma 5, $\Pr(\mathcal{L}^{\omega}_{i,j,k,l} \mid \Xi)$ is bounded by the probability that p_j and p_k belong to $\mathcal{H}(p_i, p_l, 2)$ given Ξ, and $\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathcal{L}^{\omega}_{i,j,k,l})$ is equal to the probability that for all $\gamma \neq i, j, k, l$, point p_γ is outside $\mathcal{H}(t_i, t_l, 1)$ given Ξ. Since all the points are independently and identically drawn from the uniform distribution over $\mathcal{U}$, Lemma 6 yields the result. □

We consider the three integrals $I_{x \leqslant 6}$, $I_{y \geqslant 1}$ and $I_{x>6, y<1}$ in the following subsections, and prove that each is bounded by $O\left(\frac{1}{n^3}\right)$. This will complete the proof of the upper bound of Theorem 1 since, by Lemmas 3 and 4, the expected number of $T4$-segments is less than $12\binom{n}{4}(I_{x \leqslant 6} + I_{y \geqslant 1} + I_{x>6,y<1})$.

3.2.1 $\mathbf{B}_i$ and $\mathbf{B}_l$ are close to one another

We prove here that $I_{x \leqslant 6}$ is $O(\frac{1}{n^3})$. When B_i and B_l are close to one another, the probability that there exist two other balls, B_j and B_k, defining a line tangent to B_i, B_j, B_k, B_l in that order, is small enough that we do not need to consider occlusions in order to get the bound we want.

We first bound the term $\Pr(x \leqslant \mathbf{x}_{i,l} < x + dx)$ appearing in the integral $I_{x \leqslant 6}$.

Lemma 8 $\Pr(x \leqslant \mathbf{x}_{i,l} < x + dx) \leqslant \dfrac{3x^2}{R^3} dx$.

Proof: When p_i is given, p_l must belong to a spherical shell between two spheres of center p_i and radii x and $x + dx$. The probability $\Pr(x \leqslant \mathbf{x}_{i,l} < x + dx)$, if p_i is known, is exactly the volume of the part of the spherical shell inside $\mathcal{U}$ divided by the volume of $\mathcal{U}$. The volume of the part of the spherical shell inside $\mathcal{U}$ is bounded from above by the volume of the spherical shell which is $4\pi x^2 dx$. Since the volume of $\mathcal{U}$ is $\frac{4}{3}\pi R^3$ we get the claimed bound. (The exact value of $\Pr(x \leqslant \mathbf{x}_{i,l} < x + dx)$ is actually given in [15, 22] but the above approximate bound is enough for our purposes.) □

Proposition 9 *$I_{x \leqslant 6}$ is $O(\frac{1}{n^3})$.*

Proof: Recall that (see Lemma 4)

$$I_{x\leqslant 6} = \int_{x=0}^{6} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x) \cdot \Pr(x \leqslant \mathbf{x}_{i,l} < x + dx).$$

By Lemma 7,

$$\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x) \leqslant \frac{(3x+8)^2}{R^6} \cdot \Pr(p \notin \mathcal{H}(t_i, t_l, 1) \mid \mathbf{x}_{i,l} = x,\ \mathcal{L}^{\omega}_{i,j,k,l})^{n-4} \leqslant \frac{(3x+8)^2}{R^6}.$$

It thus follows from Lemma 8 that

$$I_{x\leqslant 6} \leqslant \int_{x=0}^{6} \frac{(3x+8)^2}{R^6} \cdot \frac{3x^2}{R^3} dx = \frac{\mu^3}{n^3} \int_{x=0}^{6} 3x^2(3x+8)^2 dx = O\left(\frac{1}{n^3}\right).$$

□

3.2.2 $\mathbf{B}_i$ is entirely inside $\mathcal{U}$

For the integral $I_{y\geqslant 1}$, occlusions must be taken into account. To this aim, we bound from below the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ in the following lemma.

Lemma 10 *When $\mathcal{L}^{\omega}_{i,j,k,l}$ occurs and $\mathbf{y}_i \geqslant 1$, the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is greater than $\frac{\pi}{12}\mathbf{x}_{i,l}$.*

Proof: Let K be the ball having diameter $p_i t_i$. Note that K and p_l are both contained in $\mathcal{U}$ and in $\mathcal{H}(t_i, t_l, 1)$. The convex hull of p_l and K is thus contained in $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$, and its volume is larger than half the volume of the ball K, $\frac{\pi}{12}$, plus the volume of a cone of apex p_l, of base a disk whose boundary is a great circle of K, and of height greater than $\mathbf{x}_{i,l} - 1$. The volume of that cone is at least $\frac{1}{3}\frac{\pi}{2^2}(\mathbf{x}_{i,l} - 1) = \frac{\pi}{12}\mathbf{x}_{i,l} - \frac{\pi}{12}$. □

We now bound the probability that a tangent line segment $t_i t_l$ is not occluded by any of the other $n-4$ balls, given that the line segment $t_i t_l$ exists and the ball B_i is entirely contained in $\mathcal{U}$.

Lemma 11 $\Pr\left(p \notin \mathcal{H}(t_i, t_l, 1) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i \geqslant 1,\ \mathcal{L}^{\omega}_{i,j,k,l}\right)^{n-4} < 55 \exp\left(-\frac{\mu\, x}{16}\right).$

Proof: First notice that

$$\begin{aligned}\Pr\left(p \notin \mathcal{H}(t_i, t_l, 1) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i \geqslant 1,\quad \mathcal{L}^{\omega}_{i,j,k,l}\right)& \\ = 1 - \frac{\text{Volume of } \mathcal{H}\left(t_i, t_l, 1\right) \cap \mathcal{U}}{\text{Volume of } \mathcal{U}} &\Big|_{\mathbf{x}_{i,l}=x,\ \mathbf{y}_i \geqslant 1,\ \mathcal{L}^{\omega}_{i,j,k,l}}.\end{aligned}$$

By Lemma 10, the volume of $\mathcal{H}\left(t_i, t_l, 1\right) \cap \mathcal{U}$ is bounded from below by $\frac{\pi}{12} x$. Since the volume of $\mathcal{U}$ is $\frac{4}{3}\pi R^3$, we get

$$\Pr\left(p \notin \mathcal{H}(t_i, t_l, 1) \mid \mathbf{x}_{i,l} = x, \mathbf{y}_i \geqslant 1,\ \mathcal{L}^{\omega}_{i,j,k,l}\right)^{n-4} < \left(1 - \frac{x}{16R^3}\right)^{n-4}.$$

For any $0 \leqslant t \leqslant 1$, we have $(1-t) \leqslant e^{-t}$ thus

$$(1-t)^{n-4} \leqslant e^{-t(n-4)} = e^{-tn}e^{4t} \leqslant e^4 e^{-tn} < 55\, e^{-tn}.$$

Now $0 \leqslant x \leqslant 2R$ and $R \geqslant 1$ since B_i is entirely inside $\mathcal{U}$. Thus $0 \leqslant \frac{x}{16R^3} \leqslant \frac{1}{8R^2} \leqslant 1$ and

$$\Pr\left(p \notin \mathcal{H}(t_i, t_l, 1) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i \geqslant 1,\ \mathcal{L}^{\omega}_{i,j,k,l}\right)^{n-4} < 55 \exp\left(-\frac{n\,x}{16R^3}\right) = 55 \exp\left(-\frac{\mu\,x}{16}\right).$$

□

The following proposition now bounds the integral $I_{y \geqslant 1}$.

Proposition 12 *$I_{y\geqslant 1}$ is $O(\frac{1}{n^3})$.*

Proof: Recall that

$$I_{y\geqslant 1} = \int_{x=0}^{2R} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i \geqslant 1) \cdot \Pr(x \leqslant \mathbf{x}_{i,l} < x + dx \mid \mathbf{y}_i \geqslant 1).$$

By Lemmas 7 and 11 we have

$$\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i \geqslant 1) \leqslant \frac{(3x+8)^2}{R^6} \cdot 55 \exp\left(-\frac{\mu\,x}{16}\right).$$

Similarly as in Lemma 8 we have

$$\Pr(x \leqslant \mathbf{x}_{i,l} < x + dx \mid \mathbf{y}_i \geqslant 1) \leqslant \frac{3x^2}{R^3} dx.$$

Thus we get

$$I_{y\geqslant 1} \leqslant \int_{x=0}^{2R} \frac{(3x+8)^2}{R^6} \cdot 55 \exp\left(-\frac{\mu\,x}{16}\right) \cdot \frac{3x^2}{R^3} dx \leqslant \frac{\mu^3}{n^3} \int_{x=0}^{+\infty} 3x^2(3x+8)^2 \cdot 55 \exp\left(-\frac{\mu\,x}{16}\right) dx.$$

Changing $\frac{\mu\,x}{16}$ by z we get an integral of the kind

$$\int_0^{\infty} z^r \exp(-z) dz$$

which is bounded by a constant and thus $I_{y\geqslant 1}$ is $O\left(\frac{1}{n^3}\right)$. □

3.2.3 $\mathbf{B}_i$ and $\mathbf{B}_l$ are not close to one another and $\mathbf{B}_i$ is partially outside $\mathcal{U}$

The only remaining task is to bound the integral $I_{x>6,y<1}$. As in the previous case, we need to bound from below the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$. Here, however, the tangent $t_i t_l$ can be entirely outside $\mathcal{U}$, so the bound of Lemma 10 does not apply and a more intricate proof is needed. We need to distinguish two cases depending on the distance of segment t_it_l from O, the center of $\mathcal{U}$.

To this aim, we introduce two new types of events. For any $s \in \mathbb{R}$, let $\mathcal{F}^{\omega}_{i,j,k,l}(s)$ (resp. $\mathcal{N}^{\omega}_{i,j,k,l}(s)$) be the event that $\mathcal{L}^{\omega}_{i,j,k,l}$ occurs and the line segment t_it_l is at distance greater (resp. less) than $R+1-s$ from O. For reasons that will become clear in the proof of Lemma 15, we consider $s = y^{\frac{2}{3}}$.

The next five lemmas are used to bound the first term of the integral $I_{x>6,y<1}$.

Lemma 13 *For any random point p in $\mathcal{U}$, $\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i)$ is equal to*

$$\begin{aligned}
&\Pr\left(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i\right)\\
&\cdot \Pr\left(p \notin \mathcal{H}(t_i, t_l, 1) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4}\\
&+ \Pr\left(\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i\right)\\
&\cdot \Pr\left(p \notin \mathcal{H}(t_i, t_l, 1) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4}.
\end{aligned}$$

Proof: $\delta^{\omega}_{i,j,k,l}$ implies $\mathcal{L}^{\omega}_{i,j,k,l}$ which can be split into $\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})$, $\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})$, and the event that $\mathcal{L}^{\omega}_{i,j,k,l}$ occurs and the line segment t_it_l is at distance exactly $R+1-y^{\frac{2}{3}}$ from O. This later event occurs with probability 0, thus

$$\begin{aligned}
\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y, \mathbf{y}_l \leqslant \mathbf{y}_i) = &\\
&\Pr(\delta^{\omega}_{i,j,k,l} \cap \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i)\\
&+ \Pr(\delta^{\omega}_{i,j,k,l} \cap \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i),
\end{aligned}$$

which can be expanded into

$$\begin{aligned}
&\Pr(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i)\\
&\cdot \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}))\\
&+ \Pr(\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i)\\
&\cdot \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})).
\end{aligned}$$

When $\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})$ occurs, the probability

$$\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}))$$

is the probability that the tangent is not occluded, that is, p_γ does not belong to $\mathcal{H}(t_i, t_l, 1)$ for all the $n-4$ values of $\gamma \neq i,j,k,l$. The same argument holds for $\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})$. Since the p_γ are independent, we get the result. $\square$

In order to bound the two terms in Lemma 13,

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} \quad \text{and}$$
$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4},$$

we need to bound the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$ from below.

Lemma 14 *When* $\mathbf{x}_{i,l} \geqslant 6$, $\mathbf{y}_l \leqslant \mathbf{y}_i \leqslant 1$, $\mathcal{L}^{\omega}_{i,j,k,l}$ *occurs and segment* t_it_l *is at distance less than* $R+1-s$, $0 \leqslant s \leqslant 1$, *from the center of* $\mathcal{U}$, *then the volume of* $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$ *is larger than* $\frac{1}{6\sqrt{2}}\,(\mathbf{x}_{i,l}-5)\,s\,\sqrt{s}$.

Proof: We give here the idea of the proof; full details can be found in Appendix A. Let t be the closest point on segment t_it_l from O, and D be a unit radius disk centered at t in a plane containing O, the center of $\mathcal{U}$. We define a quadrilateral with vertices a, b, a', b' such that a and a' are the closest and the farthest points, respectively, in $D\cap\mathcal{U}$ from O, and b and b' are the points of intersection of ∂D and the perpendicular bisector of segment aa' (see Figure 2). Let v be equal to $R+1$ minus the distance from O to segment t_it_l. We prove that the convex hull of a, b, a', b' and p_l, which is included in $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$, has volume greater than $\frac{1}{6\sqrt{2}}\,(\mathbf{x}_{i,l}-5)\,\min(2\sqrt{2}, v\sqrt{v})$. It follows that, for any $0 \leqslant s \leqslant 1$, if segment t_it_l is at distance less than $R+1-s$ from O, then $v \geqslant s$ and the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$ is greater than $\frac{1}{6\sqrt{2}}\,(\mathbf{x}_{i,l}-5)\,s\,\sqrt{s}$. □

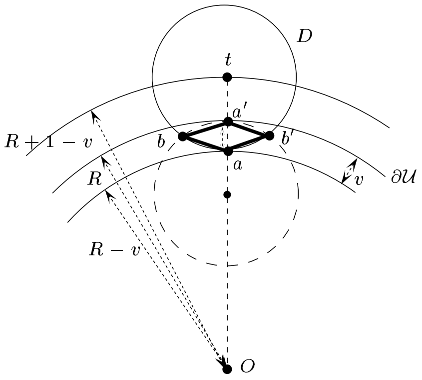


Figure 2. For the sketch of the proof of Lemma 14 ($v \in (0,1)$).

Lemma 15 *For any random point* p *in* $\mathcal{U}$, $x \geqslant 6$ *and* $0 \leqslant y \leqslant 1$,

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} < 55\exp\left(-\frac{\mu\,(x-5)\,y^2}{8\sqrt{2}\,\pi}\right)$$

and

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} < 55\exp\left(-\frac{\mu\,(x-5)\,y}{8\sqrt{2}\,\pi}\right).$$

Furthermore, if $x \geqslant 6\sqrt{R}$ then

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} < 55\exp\left(-\frac{\mu\,(x-5)}{8\sqrt{2}\,\pi}\right).$$

Proof: Let $\mathbf{x}_{i,l}=x$, $\mathbf{y}_i=y$ and suppose first that event $\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})$ occurs. Since p_i is at distance $R-y$ from O, the segment t_it_l is at distance less than $R+1-y$ from O, and thus, by Lemma 14, the volume of $\mathcal{H}\,(t_i,t_l,1)\cap\mathcal{U}$ is greater than $\frac{1}{6\sqrt{2}}\,(x-5)\,y\,\sqrt{y}$, which is bigger than $\frac{1}{6\sqrt{2}}\,(x-5)\,y^2$ since $0\leqslant y\leqslant 1$ (we bound $y\sqrt{y}$ from below by y^2 only so that we can actually compute the integral $\mathcal{I}_1$ in the proof of Proposition 20). We now follow the proof of Lemma 11, except that the volume of $\mathcal{H}\,(t_i,t_l,1)\cap\mathcal{U}$ is now bounded from below by $\frac{1}{6\sqrt{2}}\,(x-5)\,y^2$ instead of $\frac{\pi}{12}\,x$. We get

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} < 55\exp\left(-\frac{\mu\,(x-5)\,y^2}{8\sqrt{2}\,\pi}\right).$$

When $\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})$ occurs, the segment t_it_l is at distance less than $R+1-y^{\frac{2}{3}}$ from O, and thus, by Lemma 14, the volume of $\mathcal{H}\,(t_i,t_l,1)\cap\mathcal{U}$ is bounded from below by $\frac{1}{6\sqrt{2}}\,(x-5)\,y^{\frac{2}{3}}\,\sqrt{y^{\frac{2}{3}}} = \frac{1}{6\sqrt{2}}\,(x-5)\,y$. Then, as before, we get

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} < 55\exp\left(-\frac{\mu\,(x-5)\,y}{8\sqrt{2}\,\pi}\right).$$

Now, if $x\geqslant 6\sqrt{R}$, the length of the tangent t_it_l is at least $6\sqrt{R}-2$. Since $x\geqslant 6$, $R>3$ and a simple computation shows that $6\sqrt{R}-2$ is bigger than $2\sqrt{2R+1}$ which is the length of the longest line segment that may entirely lie inside $\mathcal{U}^+\backslash\mathcal{U}$. Thus $dist(O,t_it_l)\leqslant R = R+1-s$ with $s=1$ and, by Lemma 14, the volume of $\mathcal{H}(t_i,t_l,1)\cap\mathcal{U}$ is greater than $\frac{1}{6\sqrt{2}}\,(x-5)$. Then, as before, we get

$$\Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4} < 55\exp\left(-\frac{\mu\,(x-5)}{8\sqrt{2}\,\pi}\right).$$

□

Lemma 16 $\Pr(\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l\leqslant\mathbf{y}_i) \leqslant \dfrac{(3x+8)^2}{R^6}.$

Proof: The event $\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})$ occurs only if $\mathcal{L}^{\omega}_{i,j,k,l}$ occurs. The result thus follows since, by Lemmas 5 and 6, $\Pr(\mathcal{L}^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i) \leqslant \frac{(3x+8)^2}{R^6}$. □

Lemma 17 *If $y < 1$, then* $\Pr\left(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y\right) \leqslant 81\,\pi^2 \dfrac{(x+6)^2 y^2}{R^6}$.

Proof: A "far" tangent $t_i t_l$ is at distance at least $R+1-y^{\frac{2}{3}}$ from the center O of $\mathcal{U}$. Such a segment also lies in $\mathcal{H}(p_i, p_l, 1)$. Let E be the part of $\mathcal{H}(p_i, p_l, 1)$ lying outside of the sphere of radius $R+1-y^{\frac{2}{3}}$ and center O. See Figure 3 (a). Now, both p_j and p_k must be in the region inside $\mathcal{U}$ and within distance 1 from E. Denote this region by K. Then

$$\Pr\left(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i\right) \leqslant \left(\frac{\text{Volume of } K}{\text{Volume of } \mathcal{U}}\right)^2.$$

By Proposition 32, which we prove in Appendix B, the volume of K is bounded from above by $12\,\pi^2\,(x+6)\,y$, which yields the result. Here we give the intuition of the proof. Refer to Figure 3. First notice that the "length" of K is at most $x+4$. Since K is enclosed in between a sphere of radius R and one of radius $R-y^{\frac{2}{3}}$, its "height" is at most $y^{\frac{2}{3}}$. For the "width", consider Figure 3 (b) which shows a cross-section of K taken with a plane through O and perpendicular to $p_i p_l$. The "width" of K is no more than 2 times the "width" of E. The "height" of E can be bounded by some constant times $y^{\frac{2}{3}}$; thus its "width" can be bounded by some constant times $\sqrt{y^{\frac{2}{3}}} = y^{\frac{1}{3}}$. Thus, intuitively, the volume of K is smaller than $(x+4)y^{\frac{2}{3}}y^{\frac{1}{3}} = (x+4)y$, up to a constant, and the result follows. □

We now bound the two last terms of the integral $I_{x>6,y<1}$.

Lemma 18 $\Pr(y \leqslant \mathbf{y}_i < y + dy) \leqslant \dfrac{3\,dy}{R}$.

Proof: The event $(y \leqslant \mathbf{y}_i < y + dy)$ occurs only if p_i lies in the spherical shell delimited by the two spheres centered at O of radii $R-y$ and $R-y-dy$ whose volume is smaller than $4\,\pi\,R^2\,dy$. Dividing by the volume of $\mathcal{U}$ proves the result. □

Lemma 19 *For $6 \leqslant x \leqslant 2R$ and $y \leqslant 1$, we have*

$$\Pr((x \leqslant \mathbf{x}_{i,l} < x + dx) \cap (\mathbf{y}_l \leqslant \mathbf{y}_i) \mid \mathbf{y}_i = y) \leqslant \frac{6\,x\,y\,dx}{R^3}.$$

Proof: The probability $\Pr((x \leqslant \mathbf{x}_{i,l} < x + dx) \cap (\mathbf{y}_l \leqslant \mathbf{y}_i) \mid \mathbf{y}_i = y)$ is equal to the volume of the region (shown in grey in Figure 4) which is the intersection of the region in between the two spheres centered at p_i and of radii x and $x+dx$, and the region in between the two spheres centered at O and of radii R and $R-y$, divided by the volume of $\mathcal{U}$. We prove in

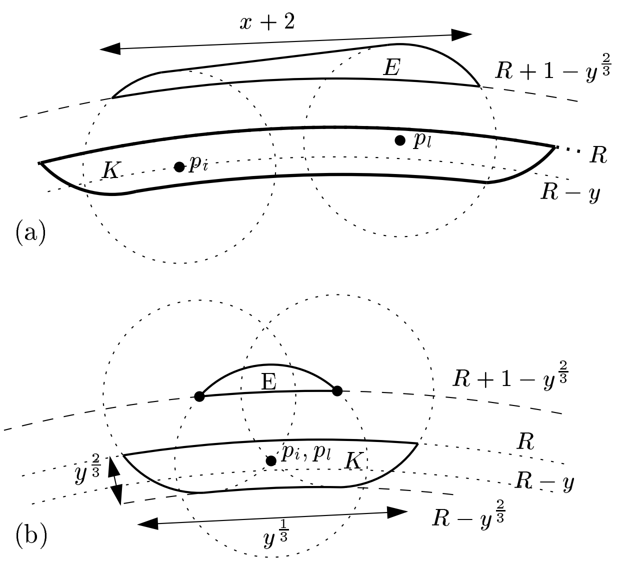


Figure 3. For the sketch of the proof of Lemma 17.

Proposition 37 in Appendix C that the volume of that region is at most $8\,\pi\,x\,y\,dx$. Roughly speaking, the volume bounded by the four spheres is at most $8\,\pi\,x\,y\,dx$ because, its "thickness" is dx, its "height" is y and its "radius" is x. Dividing by the volume of $\mathcal{U}$ proves the result. $\square$

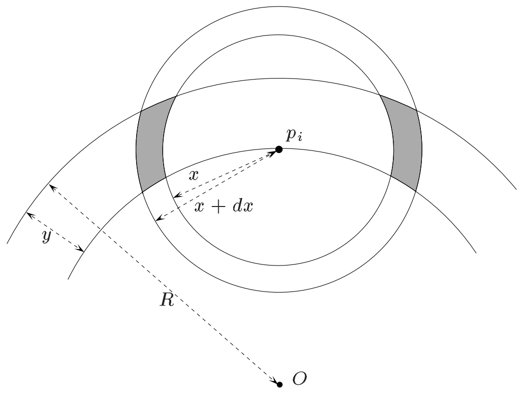


Figure 4. For the proof of Lemma 19.

We can now bound the integral $I_{x>6,y<1}$ of Lemma 4.

Proposition 20 *$I_{x>6,y<1}$ is $O\left(\frac{1}{n^3}\right)$.*

Proof: Recall that

$$I_{x>6,y<1} = \int_{x=6}^{2R}\int_{y=0}^{1} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i)$$
$$\cdot \Pr((x \leqslant \mathbf{x}_{i,l} < x+dx) \cap (\mathbf{y}_l \leqslant \mathbf{y}_i) \mid \mathbf{y}_i = y) \cdot \Pr(y \leqslant \mathbf{y}_i < y+dy).$$

By Lemmas 18 and 19, we get

$$I_{x>6,y<1} \leqslant \int_{x=6}^{2R}\int_{y=0}^{1} \Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i) \cdot \frac{6\,x\,y\,dx}{R^3}\cdot\frac{3\,dy}{R}.$$

By Lemma 13, $\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i)$ is equal to

$$\Pr\left(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i\right)$$
$$\cdot \Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4}$$
$$+ \Pr\left(\mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i\right)$$
$$\cdot \Pr\left(p \notin \mathcal{H}(t_i,t_l,1) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i,\ \mathcal{N}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}})\right)^{n-4}.$$

We split the integral at $x = 6\sqrt{R}$. When $x \geqslant 6\sqrt{R}$, the distance from O to the tangent $t_i t_l$ is less than R (see the proof of Lemma 15), which is less than $R + 1 - y^{\frac{2}{3}}$ for any y in $(0,1)$. Thus, for any $x \geqslant 6\sqrt{R}$ and $y \in (0,1)$,

$$\Pr\left(\mathcal{F}^{\omega}_{i,j,k,l}(y^{\frac{2}{3}}) \mid \mathbf{x}_{i,l}=x,\ \mathbf{y}_i=y,\ \mathbf{y}_l \leqslant \mathbf{y}_i\right) = 0.$$

It then follows from Lemmas 15, 16 and 17 that $I_{x>6,y<1} \leqslant \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$ with

$$\mathcal{I}_1 = \int_{x=6}^{6\sqrt{R}}\int_{y=0}^{1} 81\,\pi^2\frac{(x+6)^2y^2}{R^6}\cdot 55\exp\left(-\frac{\mu\,(x-5)\,y^2}{8\sqrt{2}\,\pi}\right)\cdot\frac{6\,x\,y\,dx}{R^3}\cdot\frac{3\,dy}{R},$$
$$\mathcal{I}_2 = \int_{x=6}^{6\sqrt{R}}\int_{y=0}^{1} \frac{(3x+8)^2}{R^6}\cdot 55\exp\left(-\frac{\mu\,(x-5)\,y}{8\sqrt{2}\,\pi}\right)\cdot\frac{6\,x\,y\,dx}{R^3}\cdot\frac{3\,dy}{R},$$
$$\mathcal{I}_3 = \int_{x=6\sqrt{R}}^{2R}\int_{y=0}^{1} \frac{(3x+8)^2}{R^6}\cdot 55\exp\left(-\frac{\mu\,(x-5)}{8\sqrt{2}\,\pi}\right)\cdot\frac{6\,x\,y\,dx}{R^3}\cdot\frac{3\,dy}{R}.$$

Computing these integrals with Maple gives (assuming $\mu > 0$), when R tends to infinity,

$$\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 = O\left(\frac{1}{R^9}\right).$$

Since $R^3 = n/\mu$, we get $I_{x>6,y<1} \in O(\frac{1}{n^3})$. □

We can now conclude the proof that the expected number of $T4$-segments is $O(n)$, because, by Lemmas 3, 4, and Propositions 9, 12, and 20, the expected number of $T4$-segments is smaller than

$$\sum_{(i,j,k,l)\in\mathcal{N}} \sum_{\omega=1}^{12} \left(O\left(\frac{1}{n^3}\right) + O\left(\frac{1}{n^3}\right) + O\left(\frac{1}{n^3}\right) \right) = O\left(n\right).$$

4 The expected number of $T4$-segments is at least linear

In this section, we prove that the expected number of $T4$-segments amongst n uniformly distributed unit balls is $\Omega(n)$. To do this, we bound from below the probability that four given balls have a given $T4$-segment. The key step is to give a condition on the relative positions of four unit balls that guarantees that they have exactly twelve common tangent lines. We use here the notation as defined in Section 3.1.

Lemma 21 *Let e be a real number satisfying $\frac{4\sqrt{2}}{3} < e < 2$ and let the radius R of $\mathcal{U}$ be strictly greater than e. There exists an $\epsilon > 0$ such that for any point $p \in \mathcal{U}$, there exist three balls $\Gamma_1(p)$, $\Gamma_2(p)$, $\Gamma_3(p)$ of radius ϵ contained in $\mathcal{U}$ and satisfying the following conditions:*
- *p and the centers of the $\Gamma_i(p)$ form a regular tetrahedron with edges of length e, and*
- *for any triple of points (p_1, p_2, p_3), p_i taken from $\Gamma_i(p)$, the four unit balls centered at p, p_1, p_2 and p_3 have exactly 12 distinct tangent lines.*

Proof: Macdonald, Pach and Theobald proved [14, Lemma 3] that 4 unit balls centered on the vertices of a regular tetrahedron with edges of length e, $\frac{4\sqrt{2}}{3} < e < 2$, have exactly 12 distinct real common tangent lines. Moreover, these 12 tangent lines correspond to the 12 real roots of a system of equations of degree 12, thus each tangent line corresponds to a *simple* root of that system of equations. It thus follows that for any sufficiently small perturbation of the 4 ball centers, the 4 perturbed balls still have 12 real common tangent lines. Let $\epsilon > 0$ be such that the 4 ball centers can move distance ϵ in any direction while keeping 12 distinct common tangents.

Now, for any point $p \in \mathcal{U}$, consider a regular tetrahedron with edge length e having p as a vertex and such that the other vertices are at distance at least ϵ from the boundary of $\mathcal{U}$; for example, we can choose the other three vertices on a plane perpendicular to the segment Op. Let $\Gamma_1(p)$, $\Gamma_2(p)$, and $\Gamma_3(p)$ be the balls of radius ϵ centered at the vertices, distinct from p, of that tetrahedron. By the previous reasoning, for any $q \in \Gamma_1(p)$, $r \in \Gamma_2(p)$, and $s \in \Gamma_3(p)$, the four unit balls centered at p, q, r and s have exactly twelve tangents. □

Now, by Lemma 3, the expected number of $T4$-segments is

$$\sum_{(i,j,k,l)\in\mathcal{N}} \sum_{\omega=1}^{12} \Pr(\delta^{\omega}_{i,j,k,l}).$$

Thus we only need to bound from below the probability that the event $\delta^{\omega}_{i,j,k,l}$ occurs.

Lemma 22 $\Pr(\delta^{\omega}_{i,j,k,l}) = \Omega\left(\frac{1}{n^3}\right)$.

Proof: Assume that $n > 8\mu$ so that the radius $R = \sqrt[3]{n/\mu}$ of $\mathcal{U}$ is larger than 2 and let $T(p)$ be the set $\Gamma_1(p) \times \Gamma_2(p) \times \Gamma_3(p)$ where $\Gamma_i(p)$ and e are defined as in Lemma 21. First, note that

$$\begin{aligned}\Pr(\delta^{\omega}_{i,j,k,l}) &\geqslant \Pr(\delta^{\omega}_{i,j,k,l} \cap (p_i, p_j, p_k) \in T(p_l)) \\ &= \Pr((p_i, p_j, p_k) \in T(p_l)) \cdot \Pr(\delta^{\omega}_{i,j,k,l} \mid (p_i, p_j, p_k) \in T(p_l)).\end{aligned}$$

Since $\Gamma_1(p_l)$, $\Gamma_2(p_l)$, and $\Gamma_3(p_l)$ are three balls of radius ϵ entirely contained in $\mathcal{U}$, we have

$$\Pr((p_i, p_j, p_k) \in T(p_l)) = \left(\frac{\frac{4}{3}\pi\epsilon^3}{\frac{4}{3}\pi R^3}\right)^3 = \frac{\mu^3\epsilon^9}{n^3}.$$

By Lemmas 5 and 21, the event $(\delta^{\omega}_{i,j,k,l} \mid (p_i, p_j, p_k) \in T(p_l))$ occurs if and only if the interior of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ does not contain the center of any ball. Note that the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is at most the volume of $\mathcal{H}(t_i, t_l, 1)$, which is at most $\frac{4}{3}\pi + \pi(2 + e + 2\epsilon)$ since the length of $t_i t_l$ is at most $e + 2 + 2\epsilon$. It follows that

$$\Pr(\delta^{\omega}_{i,j,k,l} \mid (p_i, p_j, p_k) \in T(p_l)) \geqslant \left(1 - \frac{\pi(\frac{4}{3} + 2 + e + 2\epsilon)}{\mathrm{Volume}(\mathcal{U})}\right)^{n-4}.$$

Since $e < 2$, we get, after some elementary calculations, that

$$\Pr(\delta^{\omega}_{i,j,k,l} \mid (p_i, p_j, p_k) \in T(p_l)) \geqslant \left(1 - \frac{(6 + 2\epsilon)\mu}{n}\right)^{n-4}. \tag{3}$$

We thus have

$$\Pr(\delta^{\omega}_{i,j,k,l}) \geqslant \frac{\mu^3\epsilon^9}{n^3}\left(1 - \frac{(6 + 2\epsilon)\mu}{n}\right)^{n-4}.$$

Since $\left(1 - \frac{(6+2\epsilon)\mu}{n}\right)^{n-4}$ tends to $e^{-(6+2\epsilon)\mu}$ when n tends to infinity, we get

$$\Pr(\delta^{\omega}_{i,j,k,l}) = \Omega\left(\frac{1}{n^3}\right).$$

$\square$

This completes the proof of the lower bound of Theorem 1 since the expected number of $T4$-segments amongst n uniformly distributed unit balls is, by Lemmas 3 and 22,

$$\sum_{(i,j,k,l) \in \mathcal{N}} \sum_{\omega=1}^{12} \Pr(\delta^{\omega}_{i,j,k,l}) = \sum_{(i,j,k,l) \in \mathcal{N}} \sum_{\omega=1}^{12} \Omega\left(\frac{1}{n^3}\right) = \Omega(n).$$

5 The expected size of the visibility complex is linear

In this section we prove Theorem 2, that the expected size of the visibility complex of a set of n uniformly distributed unit balls is linear.

We say that the balls are in general position if any k-dimensional face of the visibility complex is a connected set of maximal free segments tangent to exactly $4-k$ balls. We can assume that the balls are in general position since this occurs with probability 1. We give a bound on the expected number of k-faces, for $k = 0, \ldots, 4$.

Lemma 23 *The expected number of* 0*-faces is* $\Theta(n)$.

Proof: A 0-face of the visibility complex is a maximal free line segment tangent to 4 balls. Each maximal free line segment tangent to 4 balls contains a $T4$-segment and each $T4$-segment is contained in one maximal free line segment. Thus, by Theorem 1, the expected number of 0-faces is linear. □

To deal with the faces of dimension $k \geqslant 1$, we divide them into two classes. A k-face is *open* if it is incident to at least one $(k-1)$-face, otherwise it is *closed*. When the balls are in general position, the number of k-faces incident to a $(k-1)$-face is constant. In the proof of the following lemmas, any constant can be used. However, for completeness, we will use the exact values, but without justifying them.

Lemma 24 *The expected number of* 1*-faces is* $\Theta(n)$.

Proof: Note that a 0-face corresponds to a maximal free segment tangent to 4 balls and it is incident to those 1-faces corresponding to free segments tangent to 3 amongst those 4 balls. So, a 0-face is incident to exactly six 1-faces, which implies that the number of open 1-faces is 6 times the number of 0-faces, and is thus $\Theta(n)$ by the previous lemma.

Proving that the expected number of closed 1-faces is $O(n)$ can be done in a way very similar to the proof of the upper bound in Theorem 1. The difference is that we consider now only three balls and thus in all proofs, we forget ball B_k. We have to consider only $\binom{n}{3}$ triples of balls instead of $\binom{n}{4}$ quadruples, but we remove from the integral the probability $\Pr(p_k \in \mathcal{H}(p_i, p_l, 2) | \mathbf{x}_{i,l} = x) \leqslant \frac{3x+8}{R^3}$. Since $\frac{n}{R^3} = \mu$, this amounts to dividing the terms over which we integrate by $\mu(3x+8)$ which does not change the general shape of the integrals (a polynomial multiplied by an exponential) which are convergent. Notice that B_i, B_j, B_l and ω now define a set of segments $t_i t_l$, rather than just a single segment. However, those segments define a closed 1-face only if none of them is occluded by one of the $n-3$ remaining balls. Any particular choice of a tangent $t_i t_l$ in the 1-face will give a relevant cylinder $\mathcal{H}(t_i, t_l, 1)$ to use in the proofs. □

Lemma 25 *The expected number of* 2*-faces is* $\Theta(n)$.

Proof: Since a 1-face has five incident 2-faces, the tight linear bound on the number of 1-faces gives a tight linear bound on the number of open 2-faces. The closed case is solved similarly to the proof of the upper bound in Theorem 1. We now consider $\binom{n}{2}$ pairs of balls B_i, B_l and we remove from the integrals the probability $\Pr(p_j, p_k \in \mathcal{H}(p_i, p_l, 2) | \mathbf{x}_{i,l} = x) \leqslant \left(\frac{3x+8}{R^3}\right)^2$ which gives an $O(n)$ bound on the number of closed 2-faces. □

Lemma 26 *The expected numbers of* 3*-faces and* 4*-faces are* $\Theta(n)$*.*

Proof: A 3-face, corresponding to lines tangent to a ball, can only be closed if $n = 1$. The number of open 3-faces is linear by the fact that in general position a 2-face is incident to four 3-faces. The number of 4-faces is linear since a 3-face is incident to three 4-faces. □

6 Worst-case lower bound

We provide here a $\Omega(n^2)$ lower bound on the number of k-faces in the visibility complex. Recall that for the case of n arbitrarily sized balls, Devillers and Ramos [5] presented a simple $\Omega(n^3)$ lower bound on the number of free segments tangent to 4 balls, which is also the number of vertices in the visibility complex. Their lower bound (see Figure 5) consists of (i) $\frac{n}{3}$ balls such that the view from the origin consists of $\frac{n}{3}$ disjoint disks centered on a circle, (ii) $\frac{n}{3}$ balls such that the view from the origin consists of $\frac{n}{3}$ disks whose boundaries are concentric circles intersecting (in projection) all the disks of (i), and (iii) $\frac{n}{3}$ tiny balls centered around the origin such that from any point on these $\frac{n}{3}$ tiny balls the view of the balls in (i) and (ii) is topologically invariant. Note that finding a $\Omega(n^3)$ lower bound on the number of free segments tangent to 4 balls, amongst n balls of bounded radii, is to the best of our knowledge, open.

Proposition 27 *The number of k-faces in the visibility complex of n disjoint unit balls in* $\mathbb{R}^3$ *is* $\Omega(n^2)$ *for all k between* 0 *and* 4*.*

Proof: We first observe that the size of the visibility complex of n unit balls can trivially be quadratic by having the balls sparsely distributed in the space such that any pair of balls defines a closed 2-face.

Getting a quadratic number of free lines tangent to four balls amongst a set of n unit balls can be done by taking balls B_i centered at $(2i, 0, 0)$ for $1 \leqslant i \leqslant \frac{n}{2}$ and balls B'_j centered at $(2j, 10, 0)$ for $1 \leqslant j \leqslant \frac{n}{2}$. Then, for any i and j, the line through the points $(2i + 1, 0, 1)$ and $(2j + 1, 10, 1)$ is free and can be moved down so that it comes into contact with the four balls B_i, B_{i+1}, B'_j and B'_{j+1}. This argument proves that the number of k-faces, for $0 \leqslant k \leqslant 2$, can be quadratic.

The free segment $(2i, 1, 0)(2j, 9, 0)$ belongs to the 4-face consisting of maximum free segments with endpoints on B_i and B'_j. Thus there is a quadratic number of 4-faces. The bound also applies to 3-faces by considering lines tangent to B_i and stabbing B'_j.

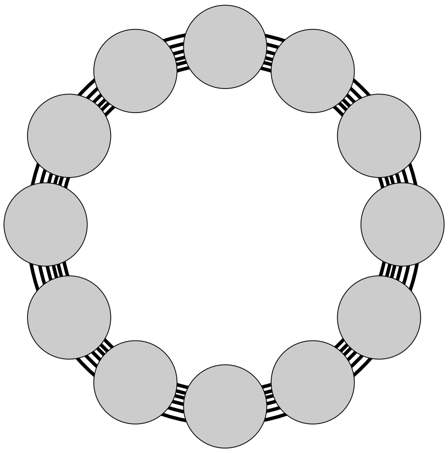


Figure 5. Quadratic view from the origin [5].

In the above construction, the balls can be pushed together (they will intersect) so that they fit inside a spherical universe of radius $\sqrt[3]{n/\mu}$ without changing the result. Note also that the above construction can be slightly perturbed to obtain the same result for a set of n unit balls, disjoint or not, with no 4 centers coplanar. $\square$

7 Generalizations

In this section we provide several generalizations of our results.

7.1 Poisson distribution

Consider a set of unit balls whose centers are drawn by a 3-D Poisson point process of parameter μ in the universe $\mathcal{U}$. By a *Poisson point process of parameter* μ *in* $\mathcal{U}$ [11], we mean that we generate X random points inside $\mathcal{U}$ so that

$$Pr(X = k) = \frac{(\mu \cdot \mathrm{Volume}(\mathcal{U}))^k \cdot \exp(-\mu \cdot \mathrm{Volume}(\mathcal{U}))}{k!} \tag{4}$$

and for any disjoint subsets M and M' of $\mathcal{U}$, the number of the points inside M and the number of points inside M' are independent random variables. Note that Equation (4) yields that the expected number of points inside $\mathcal{U}$ is $\mu \cdot \mathrm{Volume}(\mathcal{U}) = \frac{4\pi}{3}n$.

The following simple argument shows that our results extend to this distribution. Let X be the random variable representing the number of centers of unit balls generated by a

Poisson point process with parameter μ in $\mathcal{U}$, and let Y be the random variable representing the number of $T4$-segments amongst those balls. The expected number of $T4$-segments is

$$E(Y) = \sum_{k=0}^{\infty} E(Y|X = k) \cdot Pr(X = k).$$

Theorem 1 gives $E(Y|X = k) = \Theta(k)$ and

$$Pr(X = k) = \frac{(\frac{4}{3}\pi\ n)^k \cdot \exp(-\frac{4}{3}\pi\ n)}{k!}.$$

Thus

$$\begin{aligned} E(Y) &= \Theta\left(\tfrac{4}{3}\pi\ n\ \exp(-\tfrac{4}{3}\pi\ n)\ \textstyle\sum_{k=1}^{\infty} \dfrac{(\frac{4}{3}\pi\ n)^{k-1}}{(k-1)!}\right) \\ &= \Theta(n\ \exp(-\tfrac{4}{3}\pi\ n)\ \exp(\tfrac{4}{3}\pi\ n)) = \Theta(n). \end{aligned}$$

Therefore the expected number of $T4$-segments amongst n balls whose centers are generated by a Poisson point process with parameter μ in $\mathcal{U}$ is $\Theta(n)$. Similarly this bound extends to the expected size of the visibility complex.

We now investigate various models in which we change the shape of the universe or the nature of the objects.

7.2 Smooth convex universe

Our results can be generalized to the case where the universe is no longer a ball, but a homothet of a smooth convex set with homothety factor proportional to $\sqrt[3]{n}$. This can be achieved by considering the radius of curvature of the boundary of the universe, instead of R, in the proofs of the lemmas dealing with tangents outside the universe.

7.3 Other objects

Let r_{min} and r_{max} be two strictly positive real constants. In the following, we bound the expected number of $T4$-segments amongst balls whose radii vary in the interval $[r_{min}, r_{max}]$, amongst polyhedra each enclosed between two concentric balls of radii r_{min} and r_{max}, and amongst polygons each enclosed between two concentric circles of radii r_{min} and r_{max}. The centers of the concentric balls or circles are called the centers of the polyhedra or polygons, respectively. In each case a $T4$-segment is called *outer* if the centers of the two extremal objects it is tangent to are farther apart than $6r_{max}$ and are both at distance less than $2r_{max}$ from the boundary of $\mathcal{U}$. Otherwise the $T4$-segment is called *inner*.

For these models, the proof of the $\Omega(n)$ lower bound on the expected number of $T4$-segments (Section 4) generalizes directly because, for the kind of objects we consider, there always exist placements of four of them such that they admit at least one common tangent line with multiplicity one.

7.3.1 Balls of various radii

We have considered a model where all the balls have the same radius. If we allow the radii to vary in the interval $[r_{min}, r_{max}]$, then the proof of the linear upper bound on the expected number of inner $T4$-segments generalizes almost immediately by considering the volumes $\mathcal{H}(p_i, p_l, 2r_{max})$ and $\mathcal{H}(t_i, t_l, r_{min})$ instead of $\mathcal{H}(p_i, p_l, 2)$ and $\mathcal{H}(t_i, t_l, 1)$.

Section 3.2.1 generalizes immediately to prove that the expected number of $T4$-segments tangent to four balls B_i, B_j, B_k and B_l in that order such that p_i and p_l are closer to one another than $6r_{max}$ is $O(n)$. The only difficult task for extending Section 3.2.2 is the proof of the following analog of Lemma 10.

Lemma 28 *When $\mathbf{x}_{i,l} \geqslant 6r_{max}$, $\mathbf{y}_i \geqslant 2r_{max}$ and $\mathcal{L}^{\omega}_{i,j,k,l}$ occurs, the volume of $\mathcal{H}(t_i, t_l, r_{min}) \cap \mathcal{U}$ is greater than $\frac{\pi}{24} r_{min}^2(\mathbf{x}_{i,l} - 6r_{max})$.*

Proof: The proof is similar to the proof of Lemma 10. Refer to Figure 6. Let m be the midpoint of segment $t_i t_l$ and K be the sphere of diameter r_{min} centered on the point c lying on segment $t_i p_i$ at distance $\frac{1}{2} r_{min}$ from t_i. The sphere K is entirely inside $\mathcal{H}(t_i, t_l, r_{min}) \cap \mathcal{U}$, m lies in $\mathcal{H}(t_i, t_l, r_{min})$ and a straightforward computation shows that m is in $\mathcal{U}$ since t_i is in $\mathcal{U}$ at distance at least r_{max} from its boundary and t_l is at distance at most r_{max} from $\mathcal{U}$. Thus $\mathcal{H}(t_i, t_l, r_{min}) \cap \mathcal{U}$ contains the convex hull of K and m which contains the cone of apex m, of base a disk whose boundary is a great circle of K, and of height the distance from m to the center c of K. Now

$$\begin{aligned}
\mathbf{x}_{i,l} = |p_i p_l| &\leqslant |p_i c| + |cm| + |m t_l| + |t_l p_l| \\
&\leqslant r_{max} + |cm| + \frac{1}{2}|t_i t_l| + r_{max} \\
&\leqslant 2r_{max} + |cm| + \frac{1}{2}(\mathbf{x}_{i,l} + 2r_{max}).
\end{aligned}$$

Thus $|cm| \geqslant \frac{1}{2}\mathbf{x}_{i,l} - 3r_{max}$ and the volume of the cone is at least $\frac{1}{3}\pi(\frac{r_{min}}{2})^2(\frac{1}{2}\mathbf{x}_{i,l} - 3r_{max}) = \frac{\pi}{24} r_{min}^2(\mathbf{x}_{i,l} - 6r_{max})$. $\square$

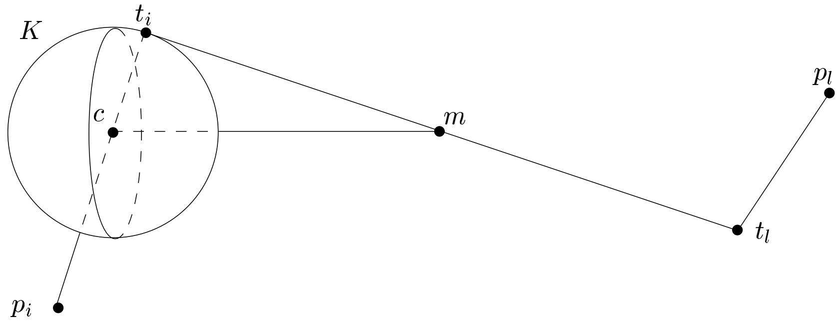


Figure 6. For the proof of Lemma 28.

The rest of Section 3.2.2 generalizes easily for proving that the expected number of $T4$-segments tangent to four balls B_i, B_j, B_k and B_l in that order such that p_i and p_l are farther apart than $6r_{max}$ and p_i is farther than $2r_{max}$ from the boundary of $\mathcal{U}$, is $O(n)$. Hence the expected number of inner $T4$-segments is $O(n)$.

Our proof cannot be extended to provide a linear upper bound on the expected number of outer $T4$-segments. This is because, if balls B_i and B_l are of radius r_{max} then a line segment $t_i t_l$ tangent to B_i and B_l might be outside $\mathcal{U}$ and at distance greater than r_{min} from its boundary. Then $\mathcal{H}(t_i, t_l, r_{min})$ does not intersect $\mathcal{U}$ and we cannot bound $\mathcal{H}(t_i, t_l, r_{min}) \cap \mathcal{U}$ from below by a positive constant as in Lemma 14, which is crucial for the proof of Lemma 15 and thus for Proposition 20.

However, by not taking into account the occlusion in the proof of Proposition 20, we get that the expected number of outer $T4$-segments is $O(n^2)$. Refer to the proof of Proposition 20 and consider $I_{x>6r_{max}, y<2r_{max}}$, the analog of $I_{x>6,y<1}$ for this case. The analogs of Lemmas 6 and 7 yield that

$$\Pr(\delta^{\omega}_{i,j,k,l} \mid \mathbf{x}_{i,l} = x,\ \mathbf{y}_i = y,\ \mathbf{y}_l \leqslant \mathbf{y}_i) \leqslant \frac{(3\,x\,r_{max}^2 + 8\,r_{max}^3)^2}{R^6}.$$

Lemma 18 still holds and we can easily prove the analog of Lemma 19. Both results imply that

$$\begin{aligned} I_{x>6r_{max}, y<2r_{max}} &\leqslant \int_{x=6r_{max}}^{2R} \int_{y=0}^{2r_{max}} \frac{(3\,x\,r_{max}^2 + 8\,r_{max}^3)^2}{R^6} \cdot \frac{6\,x\,y\,dx}{R^3} \cdot \frac{3\,dy}{R} \\ &\in O\left(\frac{1}{R^6}\right) = O\left(\frac{1}{n^2}\right). \end{aligned}$$

Hence the expected number of inner $T4$-segments is $O(n)$ and the expected number of outer $T4$-segments is $O(n^2)$. This still improves the result of Durand et al. [10] who proved a bound of $O(n^{8/3})$ for the same model.

In this section we have assumed that the sphere centers are uniformly distributed but we have made no assumption on the distribution of the radii of the spheres in the interval $[r_{min}, r_{max}]$, which are thus assumed to be worst case. The addition of some hypothesis on the radii distribution may yield better results on the number of outer T4-segments.

7.3.2 Polyhedra of bounded aspect ratio

Consider polyhedra of constant complexity, each enclosed between two concentric balls of radii r_{min} and r_{max} whose centers are uniformly distributed in $\mathcal{U}$. In such a case, as for balls of various radii, the $O(n)$ bound on the expected number of inner $T4$-segments immediately applies as well as the $O(n^2)$ bound on the expected number of outer $T4$-segments.

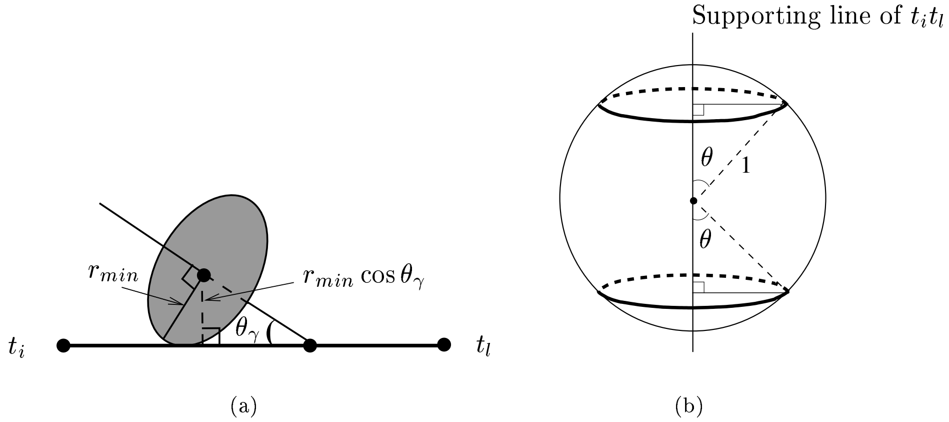


Figure 7. Illustration for the case of polygons of bounded aspect ratio.

7.3.3 Polygons of bounded aspect ratio

Our proof technique can also be generalized to non-fat 3D objects such as polygons. Consider polygons of constant complexity enclosed between two coplanar concentric circles of radii r_{min} and r_{max}, and whose centers and normals are independently chosen from the uniform distributions over $\mathbb{R}^3$ and $\mathbb{S}^2$. Let $T_1, \ldots, T_n$ be such polygons with respective normals $\mathbf{n}_1, \ldots, \mathbf{n}_n$ and centers $p_1, \ldots, p_n$.

Four polygons T_i, T_j, T_k and T_l have a common tangent line that meet them in that order only if p_j and p_k lie in $\mathcal{H}(p_i, p_l, 2r_{max})$. This implies, as in Section 3.2.1, that the expected number of $T4$-segments tangent to four polygons T_i, T_j, T_k and T_l in that order such that p_i and p_l are closer to one another than some constant, say $6r_{max}$, is $O(n)$.

When such a tangent, denoted t_it_l, exists, it is not occluded only if, for any $\gamma \neq i, j, k, l$, point p_γ does not lie in the interior of $\mathcal{H}(t_i, t_l, r_{min}\cos\theta_\gamma)$ where θ_γ denotes the angle between $\mathbf{n}_\gamma$ and the supporting line of t_it_l (see Figure 7 (a) and Lemma 5). Let γ be an integer distinct from i, j, k and l. By the Total Probability Theorem, the probability that T_γ does not occlude the tangent line segment t_it_l is bounded from above by

$$\int_{\theta=0}^{\pi/2} \Pr\left(p_\gamma \notin \mathcal{H}(t_i, t_l, r_{min}\cos\theta_\gamma) \mid \theta_\gamma = \theta\right) \cdot \Pr(\theta \leqslant \theta_\gamma < \theta + d\theta).$$

Similarly as in Lemma 28, when the tangent t_it_l exists, $\mathbf{x}_{i,l} \geqslant 6r_{max}$ and $\mathbf{y}_i \geqslant 2r_{max}$, the volume of $\mathcal{H}(t_i, t_l, r_{min}) \cap \mathcal{U}$ is greater than $\frac{\pi}{24}(r_{min}\cos\theta_\gamma)^2(\mathbf{x}_{i,l} - 6r_{max})$. Thus

$$\Pr\left(p_\gamma \notin \mathcal{H}(t_i, t_l, r_{min}\cos\theta_\gamma) \mid \theta_\gamma = \theta\right) \leqslant 1 - \frac{(r_{min}\cos\theta_\gamma)^2(\mathbf{x}_{i,l} - 6r_{max})}{32R^3}.$$

The probability that θ_γ is in between θ and $\theta + d\theta$ is $\sin\theta\, d\theta$, which corresponds to twice the area of the spherical shell between the latitudes θ and $\theta + d\theta$ on the unit sphere, divided by the area of the unit sphere (see Figure 7 (b)). Thus when p_i is at distance greater than $6r_{max}$ from p_l and at distance greater than $2r_{max}$ from the boundary of $\mathcal{U}$, the probability that T_γ does not occlude the tangent line segment t_it_l is bounded from above by

$$\int_{\theta=0}^{\pi/2}\left(1-\frac{(r_{min}\cos\theta_\gamma)^2(\mathbf{x}_{i,l}-6r_{max})}{32R^3}\right)\sin\theta\, d\theta = 1-\frac{r_{min}^2\,(\mathbf{x}_{i,l}-6r_{max})}{96\,R^3}.$$

Then, similarly as in Lemma 11, the probability that the tangent line segment t_it_l is not occluded, when p_i is at distance greater than $6r_{max}$ from p_l and at distance greater than $2r_{max}$ from the boundary of $\mathcal{U}$, is at most

$$55\exp\left(-\frac{\mu\, r_{min}^2\,(\mathbf{x}_{i,l}-6r_{max})}{96}\right).$$

We thus get the analog of Proposition 12 for the model considered here which implies that the expected number of $T4$-segments tangent to four polygons T_i, T_j, T_k and T_l in that order such that p_i and p_l are farther apart than $6r_{max}$ and p_i is farther than $2r_{max}$ from the boundary of $\mathcal{U}$ is $O(n)$.

We thus get that the expected number of inner $T4$-segments is $O(n)$. Moreover, as for balls of various radii, the expected number of outer $T4$-segments is $O(n^2)$.

8 Conclusion

In this paper, we proved that the expected number of $T4$-segments amongst n uniformly distributed unit balls in $\mathbb{R}^3$ is $\Theta(n)$, considerably improving the previously known bound of $O(n^{8/3})$ by Durand et al. [10]. We also proved that the expected size of the visibility complex of n uniformly distributed unit balls is $\Theta(n)$. Equivalently the expected number of combinatorially different visibility events amongst n uniformly distributed unit balls is $\Theta(n)$. We then proved that $\Theta(n)$ also bounds the expected number of $T4$-segments occurring not too close to the boundary of the universe for various other models such as n uniformly distributed polyhedra, or polygons, of bounded aspect ratio and constant complexity. For these models, we also provided a $O(n^2)$ bound on the expected number of all the $T4$-segments.

This paper is an attempt to analyze the average-case behavior of the size of visibility structures. The distribution models of scene objects investigated here are theoretical in nature since objects in graphics scenes are seldom distributed uniformly or by a Poisson process. However, our results are important in a context where there are few rigorous results either theoretical or experimental. They provide theoretical ground to support the empirical evidence indicating that the worst-case upper bound on the number of visibility events is largely pessimistic in practical situations. As a consequence, there is reason to believe that an output-sensitive algorithm for computing all visibility events may work in practice.

Practitioners will be concerned about the size of the constant hidden in the Θ notation. We have calculated (in the proofs of Section 3) this constant to be no larger than $2^{16}\,\mu^3 + 2^{31}\,\mu + 2^{37}\,e^{-\mu/3}\,(\mu^2 + 1/\mu^2)$. Of course this is shocking. We suppose that the constant is actually much smaller. However estimating it in practice is a difficult problem which is still to be solved. After solving this problem, an interesting experiment will be to compare the number of visibility events occurring in a realistic graphic scene with the theoretical bound for uniformly distributed objects.

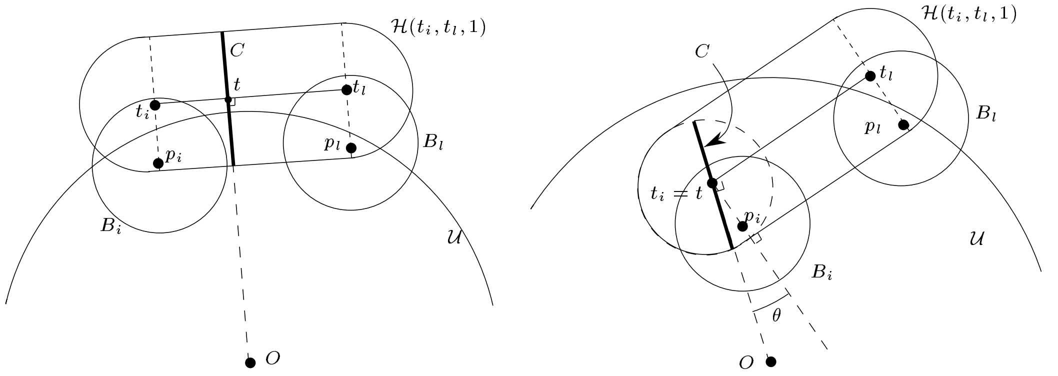
The results proved here also provide new insight on the complexity of other visibility structures. Consider for instance the aspect graph, a partition of viewpoint space into maximal connected regions by surfaces along which visibility events are observed. As explained in [19], the complexity of the aspect graph is dominated by δ^m, where δ is the degree of the surface corresponding to lines "tangent" to 3 objects and m the dimension of the viewpoint space. For a scene composed of n disjoint spheres, δ is trivially $O(n^3)$, so the aspect graph has $O(n^6)$ orthographic views and $O(n^9)$ perspective views. However the results of this paper show that the expected value of δ is $\Theta(n)$ since the expected number of families of lines tangent to three objects (related to the 1-faces of the visibility complex) is linear and the degree of each family is bounded. It would thus be interesting to get a good bound on the expected value of δ^2 and δ^3 which is related to bounding the expected value of the square and the cube of the number of combinatorially different visibility events. Note that the former would also give the standard deviation of the expected number of combinatorially different visibility events. Similar observations hold for the polyhedral case.

Appendix A. Volume of the intersection of a 3D hippodrome with a ball

Recall that $\mathcal{U}$ is a ball of radius R centered at O. Let B_i and B_l be two unit balls whose centers p_i and p_l are in $\mathcal{U}$, within distance 1 from its boundary, and distance $x \geqslant 6$ apart. Let $t_i t_l$ be a line segment tangent to B_i and B_l at its endpoints. The section is devoted to the proof of the following proposition which leads directly to Lemma 14.

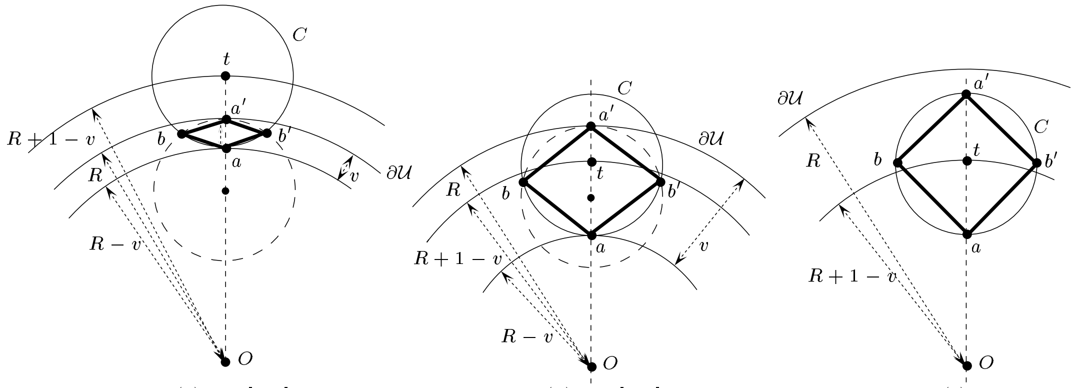
Proposition 29 *For any $0 \leqslant s \leqslant 1$ such that segment $t_i t_l$ is at distance less than $R+1-s$ from O, the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is larger than $\frac{1}{6\sqrt{2}}\,(x-5)\,s\,\sqrt{s}$.*

We proceed as follows. Let v be such that the distance from O to the segment $t_i t_l$ is $R+1-v$, and let t be the point on segment $t_i t_l$ closest to O (see Figure 8). Assume without loss of generality that t is closer to t_i than to t_l. Let C (resp. D) be the unit radius circle (resp. disk) centered at t in the plane, denoted $\mathcal{P}$, containing the vectors $\overrightarrow{Ot}$ and the cross product of $\overrightarrow{Ot}$ and $\overrightarrow{t_i t_l}$. Let θ be the angle between the plane orthogonal to $t_i t_l$ and $\mathcal{P}$. We first prove the following lemma.



(a) t is not equal to t_i or t_l; $\theta = 0$. (b) t is equal to t_i.

Figure 8. For the definition of t and C (C is shown from the side view).



(a) $v \in [0, 1]$ (b) $v \in [1, 2]$ (c) $v > 2$

Figure 9. For the definition of a, a', b, b'.

Lemma 30 *The volume of* $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ *is greater than*

$$\frac{1}{3}\min(2, \frac{v\sqrt{v}}{\sqrt{2}}) \cdot \min(\frac{x-2}{2}, (x-2)\cos\theta - 1).$$

Proof: Let a denote the closest point on C from O, a' the farthest point in $D \cap \mathcal{U}$ from O, and b and b' the two points of intersection of C and the perpendicular bisector of segment aa' (see Figure 9).

The volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is greater than the volume of the convex hull of a, b, a', b' and p_l because $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is convex and contains these five points. The volume of this polyhedron is equal to one third of the area of its base, the quadrilateral with vertices a, b, a', b', times its height, the distance from p_l to the plane $\mathcal{P}$ containing a, b, a', b'.

We first compute a lower bound on the area of the quadrilateral with vertices a, b, a', b'. If $v \leqslant 2$ (see Figure 9 (a) and (b)), then the length of aa' is equal to v, and a simple calculation gives that the length of bb' is equal to $2\sqrt{v - \frac{v^2}{4}} \geqslant \sqrt{2v}$. Thus the area of the quadrilateral a, b, a', b' is greater than $\frac{v\sqrt{v}}{\sqrt{2}}$. If $v > 2$ (see Figure 9 (c)), then C is entirely contained in $\mathcal{U}$ and the area of the quadrilateral a, b, a', b' is equal to 2. Thus, the area of the quadrilateral is at least $\min(2, \frac{v\sqrt{v}}{\sqrt{2}})$.

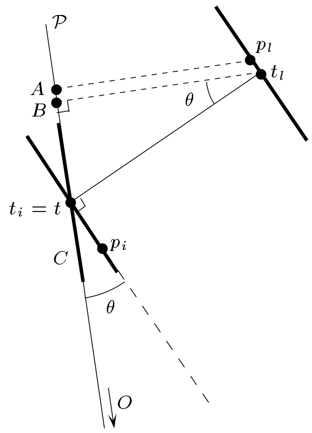


Figure 10. The height from p_l to $\mathcal{P}$ is greater than $|t_i t_l| \cos\theta - 1$.

The volume of the polyhedron is thus greater than $\frac{1}{3}\min(2, \frac{v\sqrt{v}}{\sqrt{2}})$ times the distance from p_l to the plane $\mathcal{P}$. We consider two cases.

First, suppose that t belongs to the interior of the segment $t_i t_l$ (see Figure 8 (a)). Then, the height is equal to the distance from t_l to t since p_l and a, b, a', b' belong, respectively, to the two planes, orthogonal to $t_i t_l$ and passing through t_l and t, respectively. Since t_i and t_l belong to B_i and B_l, they are at least distance $x - 2$ apart, thus t and t_l are at least distance $\frac{x-2}{2}$ apart. Thus, the height from p_l to $\mathcal{P}$ is at least $\frac{x-2}{2}$.

Second, suppose that $t = t_i$ (see Figure 8 (b)); $t \neq t_l$ since we assumed that t is closer to t_i than to t_l. Refer to Figure 10. Let A and B be the orthogonal projections of p_l and t_l onto $\mathcal{P}$, respectively. Note that the lengths of Ap_l and Bt_l are the distances from p_l and t_l to the plane $\mathcal{P}$, respectively.

Considering the triangle $\triangle Ap_l t_l$ and that the distance between t_l and p_l is 1, we obtain that $|Ap_l| \geqslant |At_l| - |t_l p_l| = |At_l| - 1$, where $|ab|$ denotes the length of segment ab. Since $A \in \mathcal{P}$ and the length of Bt_l is the distance from t_l to the plane $\mathcal{P}$, the length of At_l is greater than that of Bt_l, thus $|Ap_l| \geqslant |Bt_l| - 1$.

To bound the length of Bt_l, we now consider the triangle $\triangle Bt_l t$. The angle $\angle Bt_l t$ is the angle between the normal of the plane $\mathcal{P}$ and $t_i t_l$, that is, by definition, θ. So the length of Bt_l is the length of $t_i t_l$ times $\cos\theta$ and, since $|t_i t_l|$ is at least $x-2$, $|Bt_l|$ is greater than $(x-2)\cos\theta$. Thus the length of Ap_l is greater than $(x-2)\cos\theta - 1$.

Hence the distance from p_l to the plane $\mathcal{P}$ is greater than $\min(\frac{x-2}{2}, (x-2)\cos\theta - 1)$ and thus the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is greater than $\frac{1}{3}\min(2, \frac{v\sqrt{v}}{\sqrt{2}}) \cdot \min(\frac{x-2}{2}, (x-2)\cos\theta - 1)$. □

The following lemma bounds $\cos\theta$.

Lemma 31 *The angle θ is such that $\cos\theta \geqslant \frac{\sqrt{7}}{4}$.*

Proof: Note first that this lemma is intuitively obvious. Indeed (see Figure 8 (b)), if x is sufficiently large and if t_i is the closest point on segment $t_i t_l$ to O, then the angle between the plane supporting C and the segment $t_i t_l$ is necessarily close to $\pi/2$, which implies that θ is close to 0. We now prove the lemma.

Refer to Figure 10 and consider the triangle $\triangle Ot_i t_l$. Let $|ab|$ denote the length of segment ab. Then the law of cosines yields

$$\begin{aligned} |Ot_l|^2 &= |Ot_i|^2 + |t_i t_l|^2 - 2 \cdot |Ot_i| \cdot |t_i t_l| \cdot \cos(\tfrac{\pi}{2} + \theta) \\ &= |Ot_i|^2 + |t_i t_l|^2 + 2 \cdot |Ot_i| \cdot |t_i t_l| \cdot \sin\theta \end{aligned}$$

which gives that

$$\sin\theta = \frac{|Ot_l|^2 - |Ot_i|^2 - |t_i t_l|^2}{2 \cdot |Ot_i| \cdot |t_i t_l|}.$$

The centers p_i and p_l of balls B_i and B_l are distance $x \geqslant 6$ apart and at distance less than 1 from the boundary of $\mathcal{U}$, so $|t_i t_l| \geqslant 4$, $|Ot_i| \geqslant R-2$ and $|Ot_l| \leqslant R+1$. Hence

$$\sin\theta \leqslant \frac{(R+1)^2 - (R-2)^2 - 4^2}{2 \cdot (R-2) \cdot 4} \leqslant \frac{6(R-2)}{8(R-2)} = \frac{3}{4}.$$

Using $\cos\theta = \sqrt{1 - (\sin\theta)^2}$ proves that $\cos\theta \geqslant \frac{\sqrt{7}}{4}$. □

We can now conclude the proof of Proposition 29. For any $0 \leqslant s \leqslant 1$, if segment $t_i t_l$ is at distance $R+1-v \leqslant R+1-s$ from the center of $\mathcal{U}$, then $v \geqslant s$. By Lemma 31, $(x-2)\cos\theta - 1 \geqslant \frac{x-5}{2}$ which means that $\min(\frac{x-2}{2}, (x-2)\cos\theta - 1) \geqslant \frac{x-5}{2}$. Thus Lemma 30 gives that the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is greater than $\frac{1}{6\sqrt{2}}(x-5)\min(2\sqrt{2}, v\sqrt{v}) \geqslant \frac{1}{6\sqrt{2}}(x-5)\min(2\sqrt{2}, s\sqrt{s}) = \frac{1}{6\sqrt{2}} s\sqrt{s}(x-5)$ since $s \leqslant 1$. Hence the volume of $\mathcal{H}(t_i, t_l, 1) \cap \mathcal{U}$ is greater than $\frac{1}{6\sqrt{2}} s\sqrt{s}(x-5)$.

Appendix B. Volume of K

Recall that $\mathcal{U}$ is a ball of radius R centered at O and let p_i and p_l be two points in $\mathcal{U}$ within distance 1 of its boundary and distance x apart. Let y be a real number such that $0 \leqslant y < 1$.

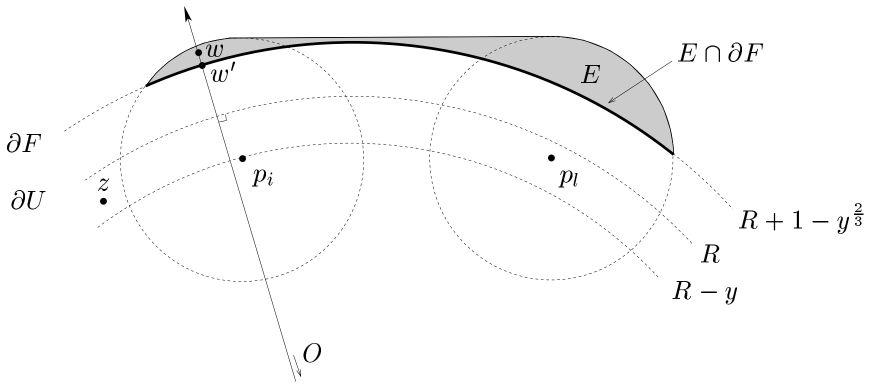


Figure 11. The part E of $\mathcal{H}(p_i, p_l, 1)$ outside F.

Let F be the open ball with center O and radius $R+1-y^{\frac{2}{3}}$ and ∂F its frontier. Let E be the part of $\mathcal{H}(p_i, p_l, 1)$ that is outside F and K be the intersection of $\mathcal{U}$ with the union of all unit balls centered on points in E (see Figure 3). This section is devoted to the proof of the following proposition used in the proof of Lemma 17.

Proposition 32 *The volume of K is bounded from above by $12\,\pi^2\,(x+6)\,y$.*

Lemma 33 *If $z \in \mathcal{U}$ is at distance less than 1 from E, then z is at distance less than 1 from $E \cap \partial F$.*

Proof: Let $z \in \mathcal{U}$ and $w \in E$ be two points at distance less than 1 and refer to Figure 11. Let w' be the point of intersection of ∂F and the ray from O through w. For any ball B centered in $\mathcal{U}$, $B \setminus F$ lies in the cone of center O and base $B \cap \partial F$. Thus $E = \mathcal{H}(p_i, p_l, 1) \setminus F$ lies in the cone of center O and base $E \cap \partial F$. Hence the ray from O through w lies in this cone and $w' \in E \cap \partial F$. On the other hand, $|zw'| \leqslant |zw|$ since $z \in F$, $w' \in \partial F$ and w lies outside F on the ray from O through w'. Thus, since $w' \in E \cap \partial F$ and $|zw| < 1$ by hypothesis, the distance from z to $E \cap \partial F$ is less than 1. $\square$

The above lemma implies that K is the union of all unit balls centered on $E \cap \partial F$. To bound the volume of K, we enclose $E \cap \partial F$ in a subset of ∂F that will be easier to deal with.

Let $B(p)$ denote the ball of unit radius centered at p. Let $\pi(p)$ be the point that maximizes (under inclusion) the intersection $\partial F \cap B(q)$ for all q on the ray from O through p. A simple computation yields that the distance between $\pi(p)$ and O is

$$R_y = \sqrt{(R+1-y^{\frac{2}{3}})^2 - 1}.$$

Thus π is the orthogonal projection onto the sphere centered at O of radius R_y. Now let $\pi'(p)$ be the point that maximizes (under inclusion) the intersection $\partial F \cap B(q)$ for all q on the radius of $\mathcal{U}$ through p (that is the part inside $\mathcal{U}$ of the ray from O through p). Similarly, π' is the orthogonal projection onto the sphere centered at O of radius

$$R' = \min(R, R_y).$$

Let G be the union of the spherical caps $\partial F \cap B(\pi'(p))$ for all p on the segment from p_i to p_l (see Figure 12). Let H denote the points of $\mathcal{U}$ at distance less than or equal to 1 from G (see Figure 13).

Lemma 34 *$K \subseteq H$.*

Proof: $E \cap \partial F$ is the union of $\partial F \cap B(p)$ for all p on the segment $p_i p_l$. Furthermore, for any such p, $\partial F \cap B(p) \subseteq \partial F \cap B(\pi'(p))$ by definition of π' since $p \in \mathcal{U}$. Thus $E \cap \partial F$ is contained in G.

By Lemma 33, K is the intersection of $\mathcal{U}$ with the union of all unit balls centered on $E \cap \partial F$. Thus K is contained in H, the union of all unit balls centered in G. □

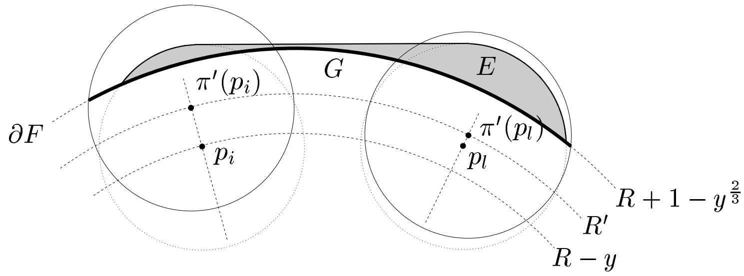


Figure 12. G, a part of ∂F enclosing $E \cap \partial F$.

To bound the volume of H from above, we first bound the area of its section by planes Π that contain O and are orthogonal to the plane, denoted (O, p_i, p_l), containing O, p_i and p_l (see Figures 13 and 14).

Lemma 35 *The area of $\Pi \cap H$ is less than $12\,\pi\,y$.*

Proof: The section of G by a plane Π is a circular arc on ∂F. If Π intersects the segment $p_i p_l$, let p denote the point of intersection, then the circular arc is the intersection of ∂F and the disk $B(\pi'(p)) \cap \Pi$ (refer to Figures 12 and 14). Otherwise, the circular arc is the intersection of ∂F and the disk $B(\pi'(p_i)) \cap \Pi$ or $B(\pi'(p_l)) \cap \Pi$ (see Figure 12). The disk has radius 1 in the former case and radius less than one in the latter case. In both cases the center of the disk is at distance R' from O. Thus the length of the circular arc $G \cap \Pi$ is

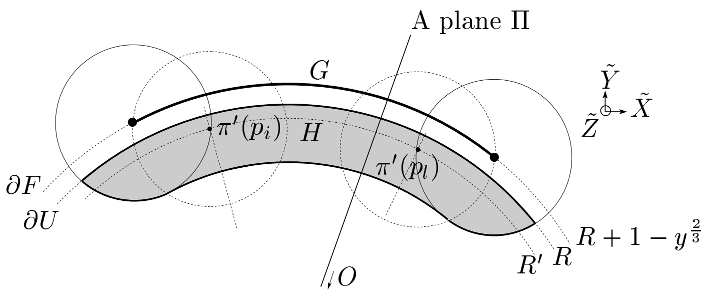


Figure 13. The region H and a plane Π.

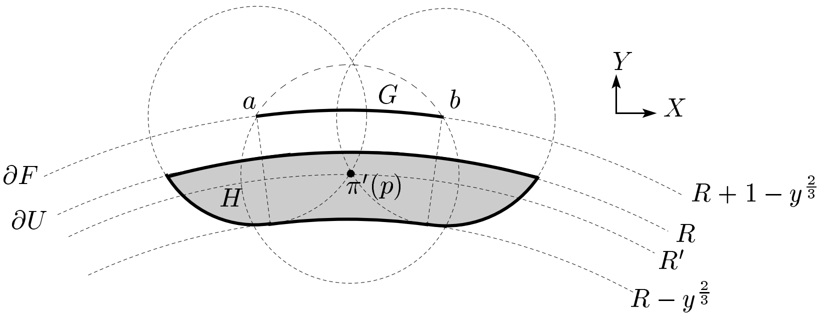


Figure 14. Section of H by a plane Π intersecting segment p_ip_l at p.

maximal if and only if Π intersects the segment p_ip_l. Thus the area of $\Pi \cap H$ is maximal if and only if Π intersects the segment p_ip_l. Hence we can assume that Π is such a plane. Let p denote its intersection with segment p_ip_l.

Let a and b denote the endpoints of $G \cap \Pi$ and refer to Figure 14. Points a and b are the intersection of ∂F and the circle in Π of radius 1 centered at $\pi'(p)$. The lines (Oa) and (Ob) split $\Pi \cap H$ into three parts, a left, a central and a right part. Symmetries with respect to the lines (Oa) and (Ob) send the left and right parts into the central one. Hence, the area of $\Pi \cap H$ is bounded by 3 times the area of its central part. This part is delimited by the two rays from O through a and b, and the two circles in Π with center O and radii R and $R - y^{\frac{2}{3}}$. So, if α denotes the length of the circular arc ab, the area A of the central part is

$$A = \frac{\alpha}{2\pi(R+1-y^{\frac{2}{3}})} \cdot \pi(R^2 - (R - y^{\frac{2}{3}})^2) = \alpha \, \frac{2Ry^{\frac{2}{3}} - y^{\frac{4}{3}}}{2(R+1-y^{\frac{2}{3}})} \leqslant \alpha \, y^{\frac{2}{3}}.$$

We now bound the length α of the arc ab. We choose an orthonormal frame $(\pi'(p), X, Y)$ in Π such that O has coordinates $(0, -R')$ (see Figure 14). Recall that a is one of the intersection points of the circle centered at $\pi'(p)$ of radius 1 and the circle centered at O of

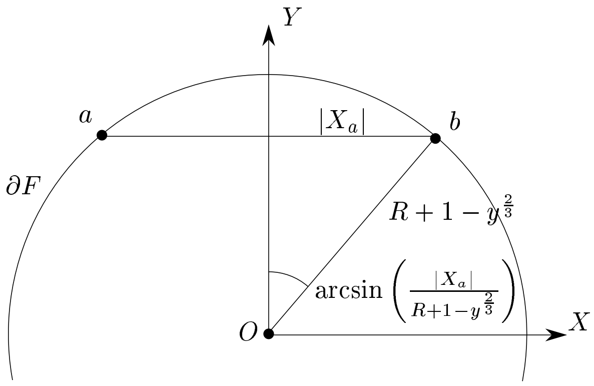


Figure 15. The length of the circular arcs ab.

radius $R+1-y^{\frac{2}{3}}$. A simple computation yields that the coordinates (X_a, Y_a) of a are equal to

$$Y_a = \frac{(R+1-y^{\frac{2}{3}})^2 - 1 - R'^2}{2R'}, \qquad |X_a| = \sqrt{1-Y_a^2}.$$

If $R' = R$, then

$$Y_a = \frac{y^{\frac{4}{3}} + 2R - 2Ry^{\frac{2}{3}} - 2y^{\frac{2}{3}}}{2R} = 1 - y^{\frac{2}{3}}(1 + \frac{2-y^{\frac{2}{3}}}{2R}) \geqslant 1 - 2y^{\frac{2}{3}}$$

which implies that

$$|X_a| \leqslant \sqrt{1-(1-2y^{\frac{2}{3}})^2} = \sqrt{4y^{\frac{2}{3}} - 4y^{\frac{4}{3}}} \leqslant 2y^{\frac{1}{3}}.$$

Now if $R' \neq R$, then $(R+1-y^{\frac{2}{3}})^2 - 1 \leqslant R^2$ by definition. Expanding this inequality yields

$$y^{\frac{4}{3}} + 2R - 2Ry^{\frac{2}{3}} - 2y^{\frac{2}{3}} \leqslant 0,$$

$$y^{\frac{2}{3}} \geqslant \frac{y^{\frac{4}{3}} + 2R}{2(R+1)} \geqslant \frac{R}{R+1} \geqslant \frac{1}{2}.$$

Thus $\sqrt{2y^{\frac{2}{3}}} \geqslant 1$ and since $|X_a| = \sqrt{1-Y_a^2} \leqslant 1$ we get $|X_a| \leqslant \sqrt{2}y^{\frac{1}{3}}$. Hence, in both cases,

$$|X_a| \leqslant 2y^{\frac{1}{3}}.$$

Thus the length of the circular arc ab is (see Figure 15)

$$\alpha = (R+1-y^{\frac{2}{3}}) \cdot 2\arcsin\left(\frac{|X_a|}{R+1-y^{\frac{2}{3}}}\right) \leqslant (R+1-y^{\frac{2}{3}}) \cdot 2\arcsin\left(\frac{2y^{\frac{1}{3}}}{R+1-y^{\frac{2}{3}}}\right).$$

A straightforward computation shows that $\arcsin(x) - \pi\, x \leqslant 0$ for any $x \in [0,1]$. Thus

$$\alpha \leqslant (R + 1 - y^{\frac{2}{3}}) \cdot 2\,\pi \frac{2y^{\frac{1}{3}}}{R + 1 - y^{\frac{2}{3}}} = 4\,\pi\, y^{\frac{1}{3}}.$$

Since the area A of the middle part is less than or equal to $\alpha\, y^{\frac{2}{3}}$,

$$A \leqslant 4\,\pi\, y^{\frac{1}{3}}\, y^{\frac{2}{3}} = 4\,\pi\, y.$$

This implies that the area of $\Pi \cap H$ is less than or equal to $12\,\pi\, y$. □

Lemma 36 *The volume of H is bounded from above by $12\,\pi^2\,(x+6)\,y$.*

Proof: We express the volume of H by an integral using spherical coordinates (r, θ, ϕ) in an orthogonal frame $(O, \tilde{X}, \tilde{Y}, \tilde{Z})$ such that the plane $(O, \tilde{X}, \tilde{Y})$ contains p_i and p_l (see Figure 13). A plane $\theta =$ constant contains the $\tilde{Z}$-axis and thus is a plane Π. Let $1_H(r, \theta, \phi)$ denote the indicator function of H; $1_H(r, \theta, \phi)$ is equal to 1 if the point of coordinates (r, θ, ϕ) belongs to H and to 0 otherwise. Then

$$\text{Volume of } H = \int_\phi \int_r \int_\theta 1_H(r, \theta, \phi) \cdot r^2 \sin\phi\, dr\, d\theta\, d\phi.$$

Since H is inside $\mathcal{U}$, $r \cdot 1_H(r, \theta, \phi) \leqslant R \cdot 1_H(r, \theta, \phi)$. Moreover $\sin\phi \leqslant 1$, thus

$$\text{Volume of } H \leqslant R \int_\theta \left(\int_\phi \int_r 1_H(r, \theta, \phi) \cdot r\, dr\, d\phi \right) d\theta.$$

The double integral in parentheses is equal to the area of the section of H by a plane $\Pi_\theta : \theta =$ constant. By Lemma 35, this area is less than $12\,\pi\, y$, which is independent of θ. Moreover the area is equal to 0 when Π_θ does not intersect H. Let $\Delta\theta$ denote the angle between the two extreme planes Π_θ that intersect H. Thus we have

$$\text{Volume of } H \leqslant R \cdot 12\,\pi\, y \cdot \Delta\theta.$$

We now bound $\Delta\theta$. Refer to Figure 16. Let Π_{θ_i} and Π_{θ_l} be the two extreme planes that intersect H. Let u_i and u_l be the two points of intersection of H with Π_{θ_i} and Π_{θ_l}, respectively; u_i and u_l lie on $\partial\mathcal{U}$. Let o_i and o_l be the two points in G at distance 1 from u_i and u_l, respectively. $\pi'(p_i)$ and $\pi'(p_l)$ are at distance 1 from o_i and o_l, respectively.

The angle between the two extreme planes Π_{θ_i} and Π_{θ_l} is, as before,

$$\Delta\theta = 2 \arcsin \frac{|u_i u_l|/2}{R} \leqslant 2\,\pi \frac{|u_i u_l|/2}{R} = \pi \frac{|u_i u_l|}{R}.$$

Now we bound $|u_i u_l|$ by the length of the polygonal line shown in Figure 16.

$$\begin{aligned} |u_i u_l| &\leqslant |u_i o_i| + |o_i \pi'(p_i)| + |\pi'(p_i) p_i| + |p_i p_l| + |p_l \pi'(p_l)| + |\pi'(p_l) o_l| + |o_l u_l| \\ &= 1 + 1 + |\pi'(p_i) p_i| + x + |p_l \pi'(p_l)| + 1 + 1. \end{aligned}$$

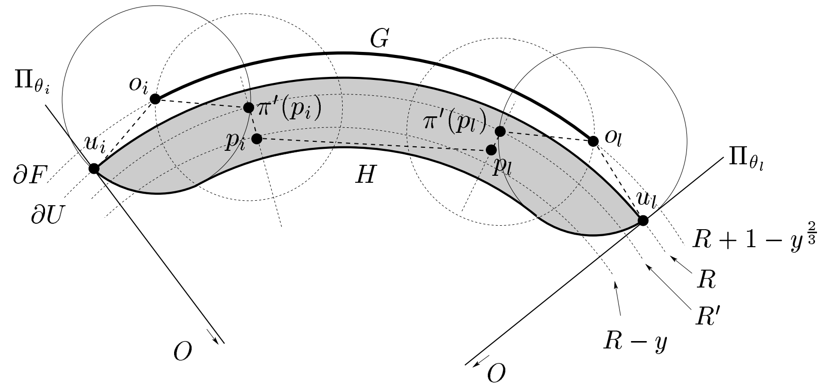


Figure 16. For the computing a bound on $\Delta\theta$.

We show that $|\pi'(p_i)p_i|$ and $|p_l\pi'(p_l)|$ are less than 1. $\pi'(p_i)$ is inside $\mathcal{U}$ at distance less than 1 from ∂F which lies outside $\mathcal{U}$. Thus $\pi'(p_i)$ is inside $\mathcal{U}$ at distance less than 1 from its frontier. Point p_i is also inside $\mathcal{U}$ at distance less than 1 from its frontier. Since p_i and $\pi'(p_i)$ are on the same ray starting from O, they are at distance less than 1 apart. Similarly for $\pi'(p_l)$ and p_l. Hence

$$\Delta\theta \leqslant \pi\frac{|u_iu_l|}{R} \leqslant \pi\frac{x+6}{R}.$$

Therefore

$$\text{Volume of } H \leqslant R\cdot 12\,\pi\, y\cdot\Delta\theta \leqslant 12\,\pi^2\, y\,(x+6).$$

□

Proposition 32 follows from Lemmas 34 and 36.

Appendix C. Volume of the intersection of two spherical shells

We prove in this section the following proposition used in the proof of Lemma 19.

Proposition 37 *Let $R>0$, $x\in[6,2R]$, $y\in[0,1]$ and p be a point at distance $R-y$ from O. The volume of the intersection of the region in between the two spheres centered at p and of radii x and $x+dx$, and the region in between the two spheres centered at O and of radii R and $R-y$ (see Figure 17) is bounded from above by $8\,\pi\,x\,y\,dx$.*

Proof: Define the balls B_1 with center O and radius R, B_2 with center O and radius $R-y$, B_3 with center p and radius x and finally B_4 with center p and radius $x+dx$. Let $\mathcal{V}$ denote

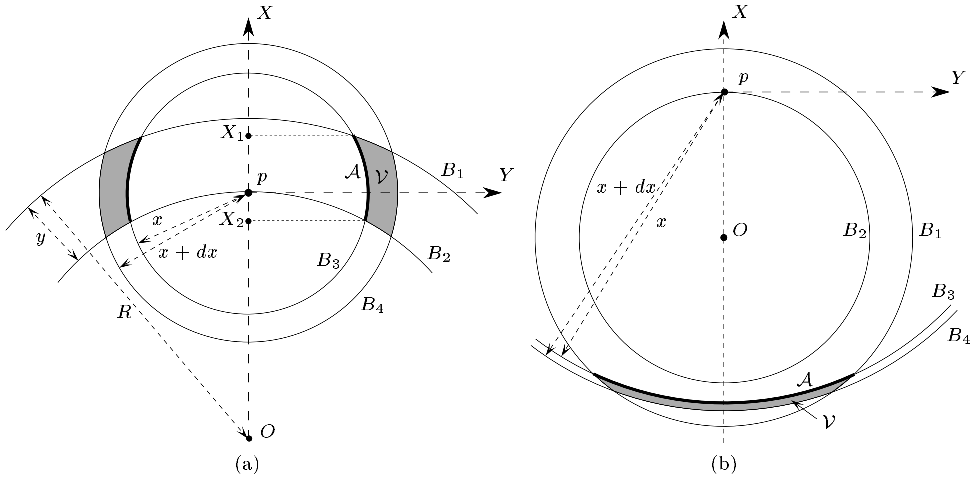


Figure 17. For the proof of Proposition 37.

the intersection of $(B_1 \setminus B_2)$ and $(B_4 \setminus B_3)$. We prove that the volume of $\mathcal{V}$ is less than $8\,\pi\,x\,y\,dx$.

Since dx is infinitesimally small, the volume of $\mathcal{V}$ is $\mathcal{A}dx$ where $\mathcal{A}$ is the area of the intersection of the sphere ∂B_3 with $B_1 \setminus B_2$.

Let (p, X, Y, Z) be an orthogonal reference frame whose center is p and whose X-axis is oriented along $\overrightarrow{Op}$ (see Figure 17). Notice that all spheres are centered on that axis. Let C_1 (resp. C_2, C_3) denote the circle that is the boundary of the intersection of B_1 (resp. B_2, B_3) and the plane (p, X, Y) in which Figure 17 is drawn. The equations of these circles are, in the frame (p, X, Y),

$$\begin{array}{rrcl} C_1: & (X+R-y)^2+Y^2 & = & R^2, \\ C_2: & (X+R-y)^2+Y^2 & = & (R-y)^2, \\ C_3: & X^2+Y^2 & = & x^2. \end{array}$$

Since C_3 is centered at a point on C_2 and has radius $x \geqslant 6 > 1 \geqslant y$, C_3 intersects or encloses C_1 and C_2. In fact, C_3 intersects or encloses C_1 and C_2 in one of the three following ways.

Case 1: If $6 \leqslant x \leqslant 2R - 2y$ then C_3 intersects both C_1 and C_2 (see Figure 17 (a)).
Case 2: If $2R - 2y < x \leqslant 2R - y$ then C_3 intersects C_1 and encloses C_2 (see Figure 17 (b)).
Case 3: If $2R - y < x$ then C_3 encloses both C_1 and C_2. In that case, $\mathcal{V}$ is empty and the volume is 0.

In the first case, let X_1 (resp. X_2) be the abcissa of the points of intersection of circles C_1 (resp. C_2) and C_3. Note that $X_1 \geqslant X_2$ and their values can be computed directly from

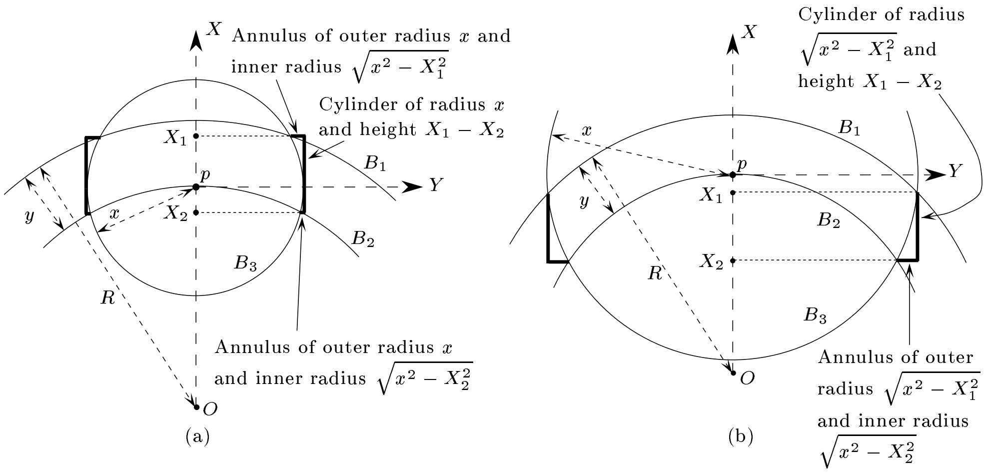


Figure 18. For the proof of Proposition 37, case 1.

the equations of the circles C_1, C_2 and C_3:

$$X_1 = \frac{R^2 - x^2 - (R-y)^2}{2R - 2y}, \qquad X_2 = \frac{-x^2}{2R-2y}.$$

Using the fact that $y \leqslant x \leqslant 2R - 2y$ we get

$$\begin{aligned} X_1 - X_2 &= \frac{y(2R-y)}{2R-2y} = y\left(1 + \frac{y}{2R-2y}\right) \leqslant 2y, \\ -X_1 - X_2 &= \frac{2x^2 - y(2R-y)}{2R-2y} \leqslant 2x\frac{x}{2R-2y} \leqslant 2x. \end{aligned}$$

We now bound from above the area $\mathcal{A}$ of the surface $\partial B_3 \cap (B_1 \setminus B_2)$ by the area of a larger surface which depends on the sign of X_1. If $X_1 \geqslant 0$, the surface consists of a cylinder of axis the X-axis, of radius x and height $X_1 - X_2$, and of two annuli in the planes $X = X_1$ and $X = X_2$, of inner radius $\sqrt{x^2 - X_1^2}$ and $\sqrt{x^2 - X_2^2}$, respectively, and outer radius x (see Figure 18 (a)). If $X_1 \leqslant 0$, the surface consists of a cylinder of axis the X-axis, of radius $\sqrt{x^2 - X_1^2}$ and height $X_1 - X_2$, and of an annulus in the plane $X = X_2$, of inner radius $\sqrt{x^2 - X_2^2}$ and outer radius $\sqrt{x^2 - X_1^2}$ (see Figure 18 (b)). In both cases that surface is larger than $\partial B_3 \cap (B_1 \setminus B_2)$ by convexity.

If $X_1 \geqslant 0$, the area of the cylinder is $2\pi x(X_1 - X_2) \leqslant 4\pi xy$ and the area of the annuli are $\pi x^2 - \pi(x^2 - X_i^2) = \pi X_i^2$, $i = 1, 2$. Since $X_1 \geqslant 0$, $X_1 \leqslant y \leqslant x$ and thus $\pi X_1^2 \leqslant \pi xy$. We also have from the expression of X_1 that $R^2 - x^2 - (R-y)^2 \geqslant 0$ and thus $x^2 \leqslant y(2R - y)$.

Thus

$$\pi X_2^2 = \pi \frac{x^2}{2R-2y}\frac{x^2}{2R-2y} \leqslant \pi x \frac{x}{2R-2y} y \frac{2R-y}{2R-2y} = \pi xy \frac{x}{2R-2y}\left(1+\frac{y}{2R-2y}\right).$$

It thus follows from $y \leqslant x \leqslant 2R-2y$ that $\pi X_2^2 \leqslant 2\pi xy$. Hence $\mathcal{A} \leqslant 7\pi xy$.

If $X_1 \leqslant 0$, the area of the cylinder is $2\pi\sqrt{x^2 - X_1^2}(X_1 - X_2) \leqslant 2\pi x(2y)$ and the area of the annulus is $\pi(x^2 - X_1^2) - \pi(x^2 - X_2^2) = \pi(X_1 - X_2)(-X_2 - X_1) \leqslant 4\pi xy$. Thus $\mathcal{A} \leqslant 8\pi xy$.

Consider now the second case $2R - 2y < x \leqslant 2R - y$ (see Figure 17 (b)). For a fixed value of y, $\mathcal{A}$ is the area of a spherical cap whose perimeter and curvature decreases as x increases. Thus $\mathcal{A}$ is a decreasing function of x. Since the bound $\mathcal{A} \leqslant 8\pi xy$ is valid for $x = 2R - 2y$ and $8\pi xy$ is an increasing function of x, $\mathcal{A} \leqslant 8\pi xy$ for any $x \geqslant 2R - 2y$. □

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