Deterministic tilings, periodicity and undecidability

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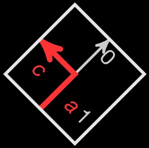
8 avril 2014



In this talk

We meet:

- tilings by Wang tiles
- deterministic tilesets
- Turing machine simulations
- undecidable tiling problems

We prove the **undecidability of the Periodic Domino Problem** when the input is restricted to **4-way deterministic** tilesets.

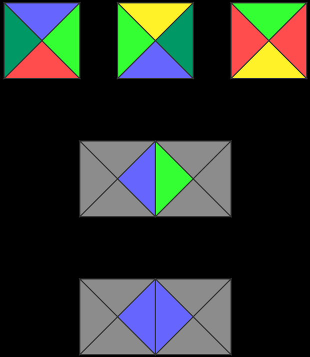
1. Tilings

Tilings by Wang tiles

A **Wang tile** is an oriented (no rotations allowed) unit square tile carrying a **color on each side**.

A **tileset** τ is a finite set of Wang tiles.

A **tiling** $c : \mathbb{Z}^2 \longrightarrow \tau$ associates a tile to each cell of the discrete plane $\mathbb{Z}^2$ in such a way that the colors of the common sides of neighboring tiles match.



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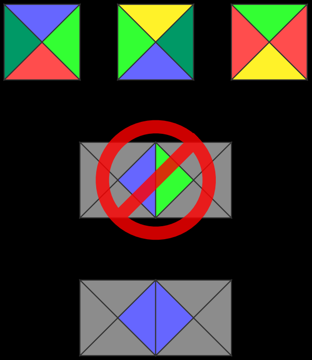
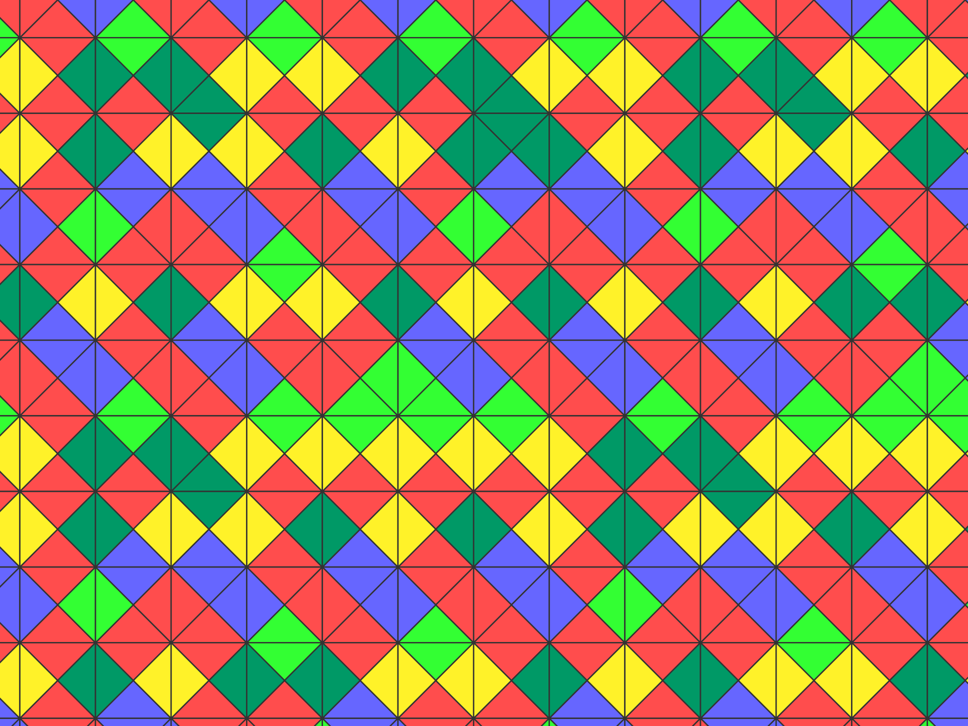




Domino Problem (1/2)

Domino Problem [Wang 61]. Given a Wang tileset, decide whether it tiles the discrete plane.

Simple facts.

1. If a tileset admits a periodic tiling, then it admits a bi-periodic tiling.
2. The Domino Problem is co-recursively enumerable.
3. Deciding whether a tileset admits a periodic tiling is recursively enumerable.

Wang's conjecture. If a tileset tiles the plane, then it admits a periodic tiling. Hence **DP** is decidable.

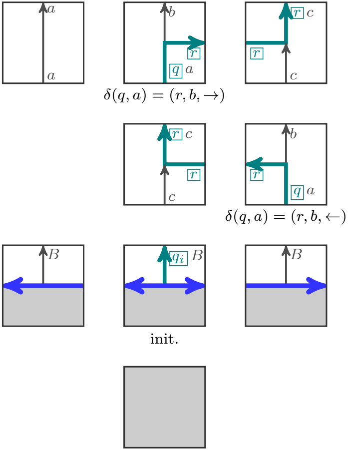
Actually... There exists an **aperiodic** tileset **[Berger 64]** (103 tuiles).

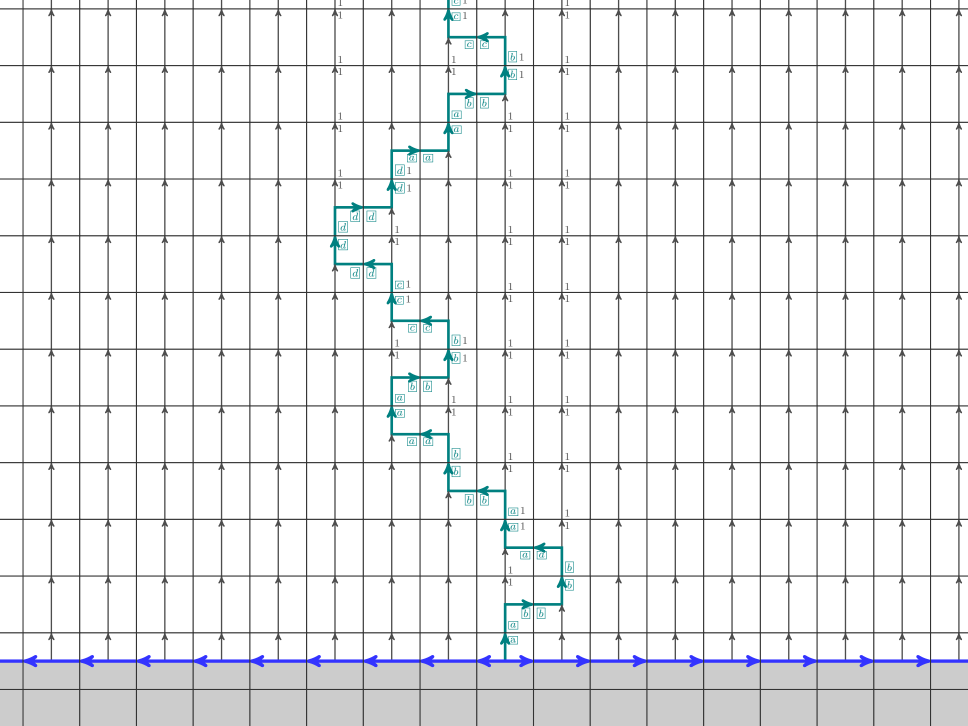
Domino Problem with a seed tile

DP with a seed tile. Given a tileset with an identified *seed tile*, decide whether it admits a tiling containing the seed tile.

Undecidable. Use the seed tile to properly initialize a Turing computation.

There exists a tiling containing the seed tile ⇔ the machine does not halt from the empty tape.

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Domino Problem (2/2)

Theorem [Berger 64]. **DP** is undecidable.

Proof simplified later by **[Robinson 71]**.

The existence of **aperiodic** tilesets and this result justify the definition of a natural variant of the Domino Problem…

Periodic Domino Problem

Periodic Domino Problem. Given a tileset, decide whether it admits a **periodic** tiling.

[Gurevich-Koryakov 72] modifies Berger's construction to deal with this problem.

Theorem [Gurevich-Koryakov 72]. **PDP** is undecidable.

Or, equivalently:

Theorem. Periodic and aperiodic tilesets are recursively inseparable.

This can also easily be proved by a simple modification of Robinson's construction.

But also…

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But also… more on that later.

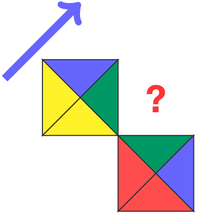
2. Determinism

Deterministic tilesets

Introduced by J. Kari in 1991 to prove the undecidability of the nilpotency problem for 1D cellular automata.

Notations: *NW* for North-West, *SE* pour South-East…

Deterministic tileset. A tileset τ is **NE-deterministic** if for any pair of tiles $(t_W, t_S) \in \tau^2$, there exists **at most one** tile t compatible to the west with t_W and to the south with t_S.



Partial local map.

$$f : \tau^2 \longrightarrow \tau$$

We symmetrically define **{NW,SE,SW}-determinism**.

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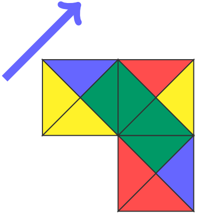


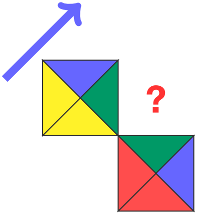
Partial local map.

$$f : \tau^2 \to \tau$$

We symmetrically define **{NW,SE,SW}-determinism**.

Deterministic tilesets

Bi-determinism. A tileset is **bi-deterministic** if it is simultaneously deterministic in **two opposite directions**: NE & SW, or NW & SE.



4-way determinism. A tileset is **4-way deterministic** if it is simultaneously deterministic in the **4 directions** NE, NW, SW and SE.

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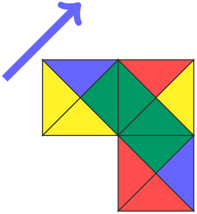

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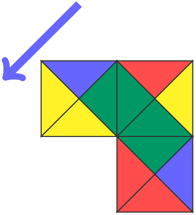

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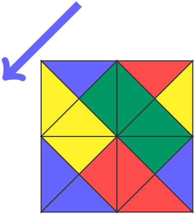

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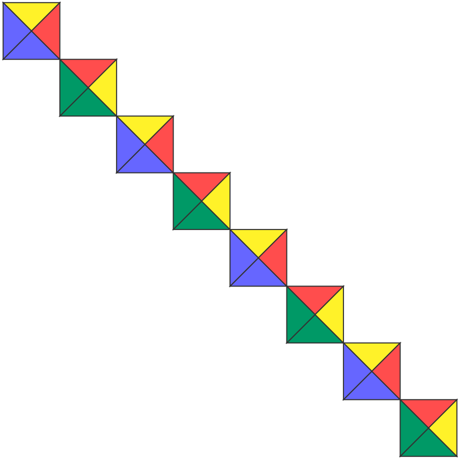

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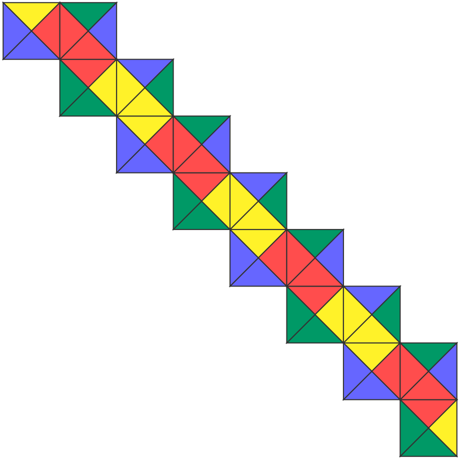

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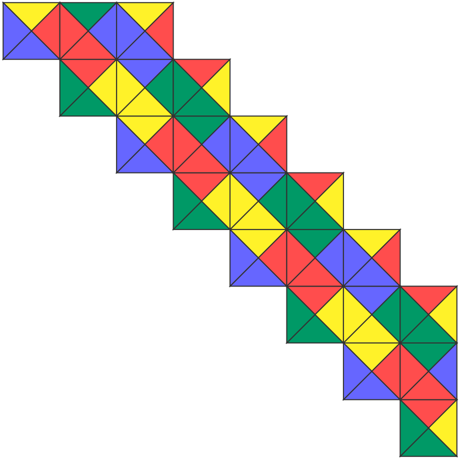

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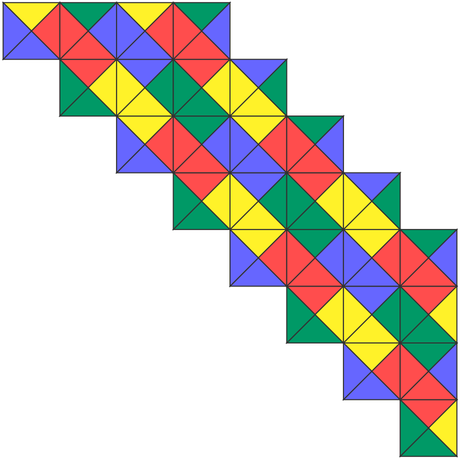

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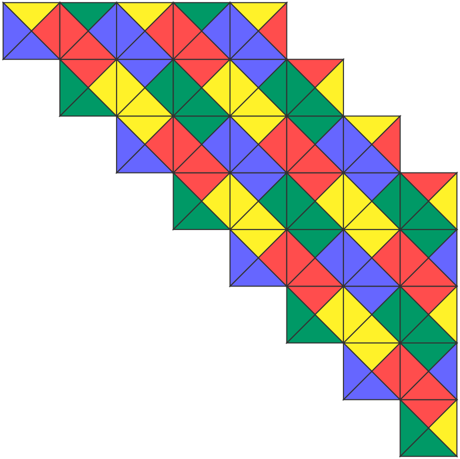

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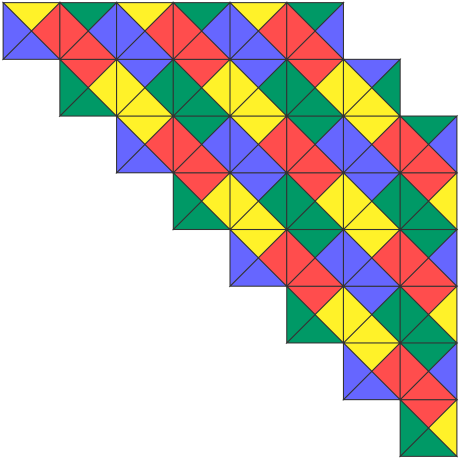

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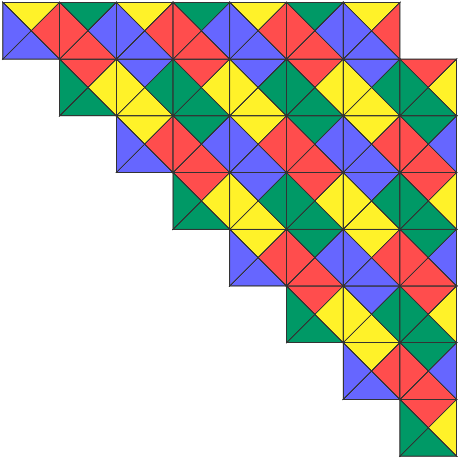

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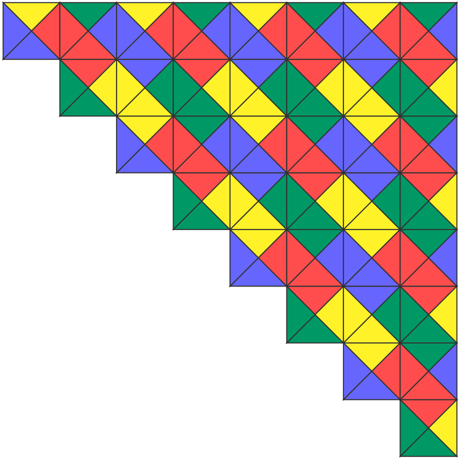

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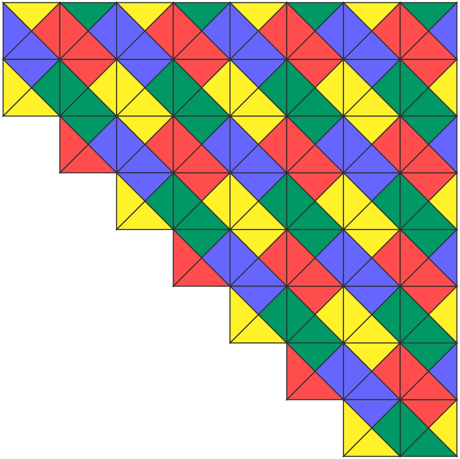

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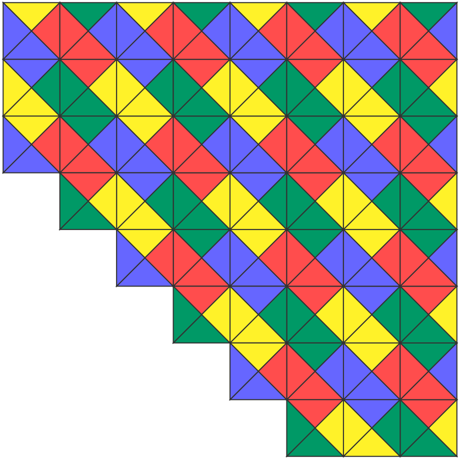

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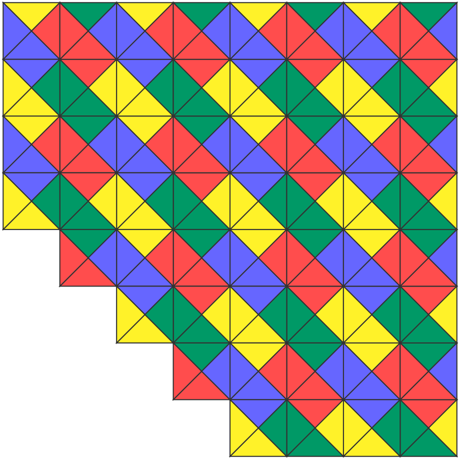

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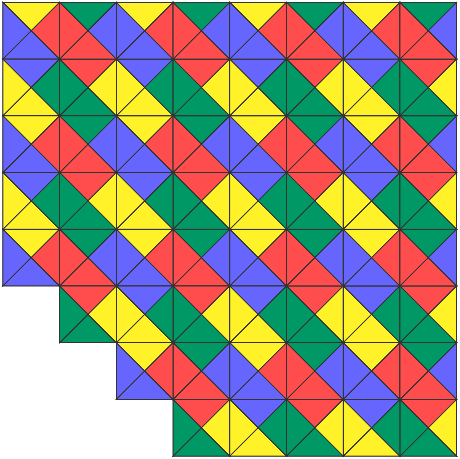

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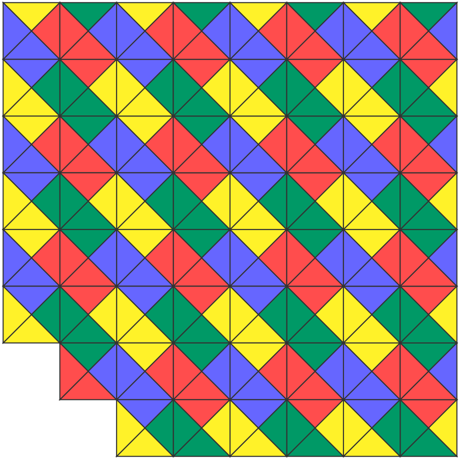

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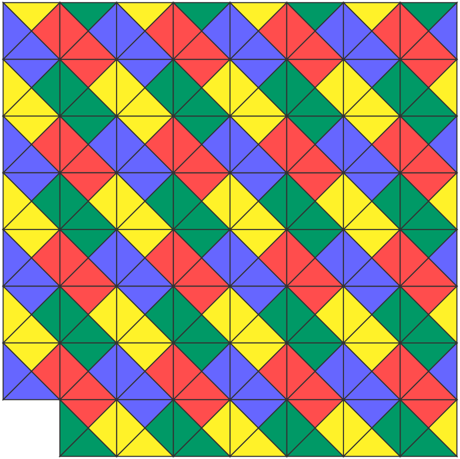

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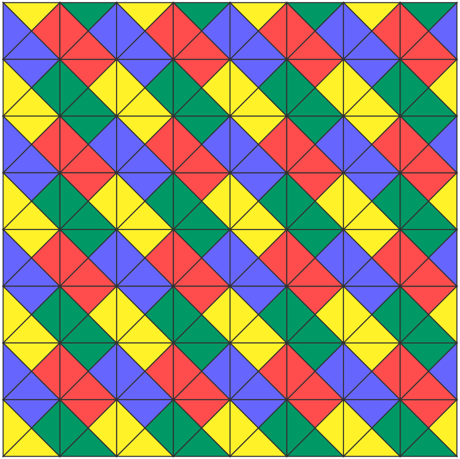

Deterministic tilesets: a short history

[Kari 91] introduced a (bi-)determinization of **[Robinson 71]** to treat the nilpotency problem for cellular automata in dimension 1 (**Nil1D**).

Theorem [Kari 91]. **Nil1D** is undecidable.

Theorem [Kari 91]. There exist some **(bi-)deterministic** aperiodic tilesets.

N.B. The 16 Wang tiles derived from Ammann's geometric tiles are bi-deterministic.

Theorem [Kari 91]. **DP** remains undecidable for (one-way) **deterministic** tilesets.

Deterministic tilesets: a **revised** history

[Aanderaa-Lewis 74] embedded the coding of two-dimensional Wang tilings into one-dimensional double shifts, allowing this to be coded back into deterministic Wang tilesets or one-dimensional cellular automata.

Theorem [Aanderaa-Lewis 74]. **Nil1D** is undecidable.

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Deterministic tilesets: a short history

[Kari-Papasoglu 99] builds a strong determinization of **[Robinson 71]**.

Theorem [Kari-Papasoglu 99]. There exist some **4-way** deterministic aperiodic tilesets.

[Lukkarila 09] introduces a 4-way determinization of **[Robinson, 1971] + Turing computation**.

Theorem [Lukkarila 09]. **DP** remains undecidable for **4-way** deterministic tilesets.

3. The 4-way deterministic **PDP**

Back to **PDP**

What about the deterministic setting? Mazoyer and Rapaport considered this problem in the deterministic case to prove the undecidability of the nilpotency problem **over periodic configurations** of one-dimensional cellular automata.

Theorem [Mazoyer-Rapaport 99]. **Nil1D** over **periodic** configurations is undecidable.

Theorem [Mazoyer-Rapaport 99]. **PDP** remains undecidable for (one-way) **deterministic** tilesets.

The proof is a slightly technical modification of **[Kari 91]**.

Result of the day

Let us prove the following.

Theorem. PDP remains undecidable for **4-way** deterministic tilesets.

Natural method. Modify a construction for **DP**.

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[Jeandel 10] introduced a new proof method for the undecidability of **PDP**, which will inspire us.

Instead of modifying a construction for **DP**, the proof only takes as input an **arbitrary** aperiodic tileset.

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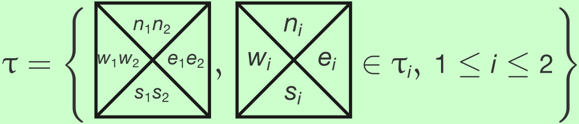
Instead of modifying a construction for **DP**, the proof only takes as input an **arbitrary** aperiodic tileset.

We will also build upon the 4-way deterministic **Turing machine simulation** from **[Lukkarila 09]**.

Preliminaries: Transformations (1/3)

Product. Given two tilesets $\tau_1 \subseteq \mathcal{C}_1^4$ and $\tau_2 \subseteq \mathcal{C}_2^4$, the **product** tileset $\tau = \tau_1 \times \tau_2$ is



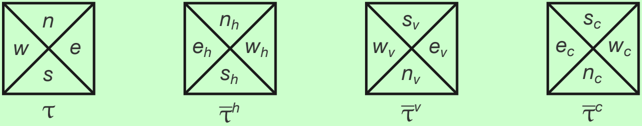
over the set of colors $\mathcal{C}_1 \times \mathcal{C}_2$.

We interpret it as a **two-layered** tileset whose tiles hold a tile of τ_1 on the first layer and a tile of τ_2 on the second layer (with a local matching condition requiring that the matching conditions of both layers are verified).

If τ_1 and τ_2 are **4-way deterministic**, then $\tau_1 \times \tau_2$ is also 4-way deterministic.

Preliminaries: Transformations (2/3)

Disjoint mirrors. Disjoint tilesets by duplication of colors.



If a tileset is **4-way deterministic**, any of its disjoint mirrors is also 4-way deterministic (some determinism directions are interchanged).

Any of the (disjoint) unions of a tileset with some of its disjoint mirrors (e.g. $\tau \sqcup \overline{\tau}^h \sqcup \overline{\tau}^v \sqcup \overline{\tau}^c$) is also 4-way deterministic.

Preliminaries: Transformations (3/3)

Grouping. Given a tileset τ, its 2×2 **grouping** $\tau^{2\times 2}$ is

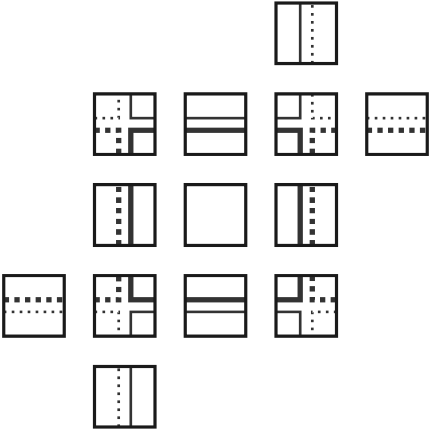
$$\tau^{2\times 2} = \left\{ \boxed{\begin{smallmatrix} & ab & \\ ac & & bd \\ & cd & \end{smallmatrix}} \in (\tau^2)^4,\ \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array} \text{ is a valid } 2 \times 2 \text{ pattern by } \tau \right\}$$

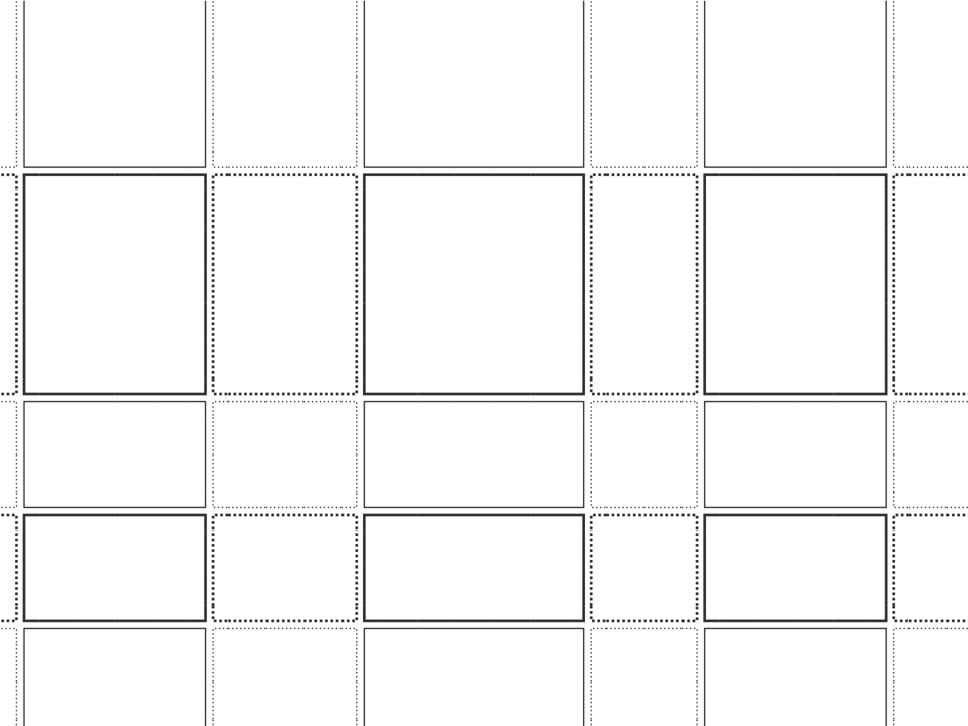
over the set of colors τ^2.

The tileset $\tau^{2\times 2}$ is a coding of 2×2 patterns by τ with the local matching condition that two horizontally (resp. vertically) adjacent patterns overlap on one column (resp. row).

If τ is **4-way deterministic**, then $\tau^{2\times 2}$ is also 4-way deterministic.

Layer 1: Grid





Layers 2 and 3: Mirrors (1/2)

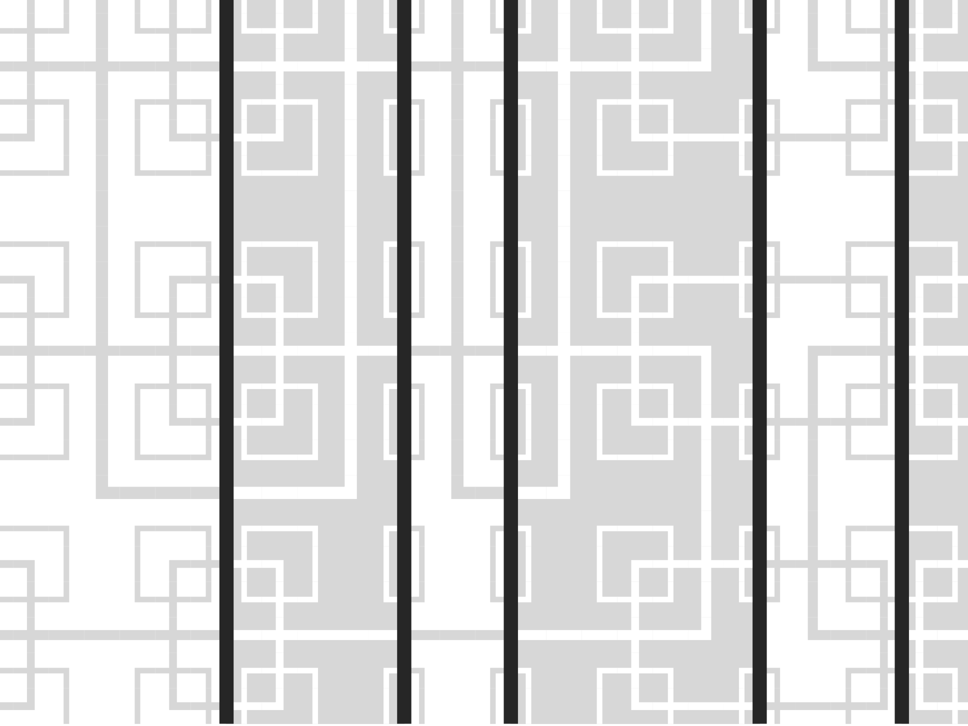
We consider τ_a an arbitrary **4-way deterministic aperiodic** tileset over the set of colors $\mathcal{C}$.

Let $\overline{\tau_a}^h$ be its **horizontal mirror tileset** over the set of colors $\mathcal{C}_h = \{x_h,\ x \in \mathcal{C}\}$ (disjoint copy of $\mathcal{C}$).

For all $x \in \mathcal{C}$, we add the following tiles to $\tau_a \sqcup \overline{\tau_a}^h$.

Lemma. The built tileset is **4-way** deterministic.

Lemma. Every horizontally periodic tiling contains a vertical mirror line. Hence, by periodicity, it contains infinitely many such lines (with bounded distance between two consecutive lines).



Layers 2 and 3: Mirrors (2/2)

We add a **third layer**, similarly to what we have done for the second layer but with **horizontal mirror lines**.

We **synchronize** the lines of the grid of the first layer with the mirror lines of the second and third layers.

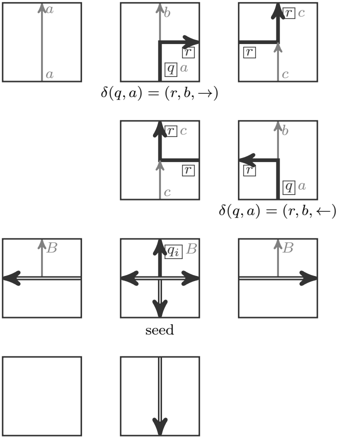
Lemma. The built tileset is **4-way** deterministic.

Lemma. Every bi-periodic tiling contains a grid with bounded rectangle size on its first layer.

Layer 4: Turing computation (1/4)

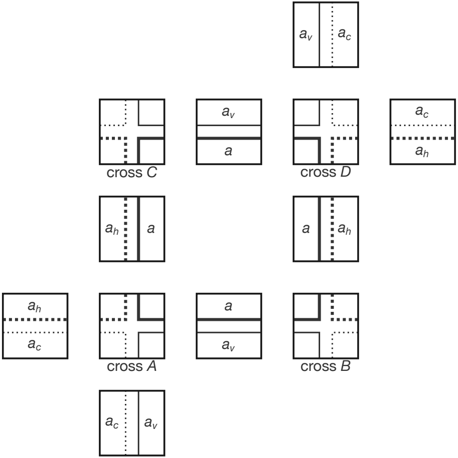
Given a Turing machine *M*, **[Lukkarila 09]** introduces a **4-way** deterministic tileset τ_M to prove the undecidability of the Domino Problem **with a seed tile** in the 4-way deterministic setting.

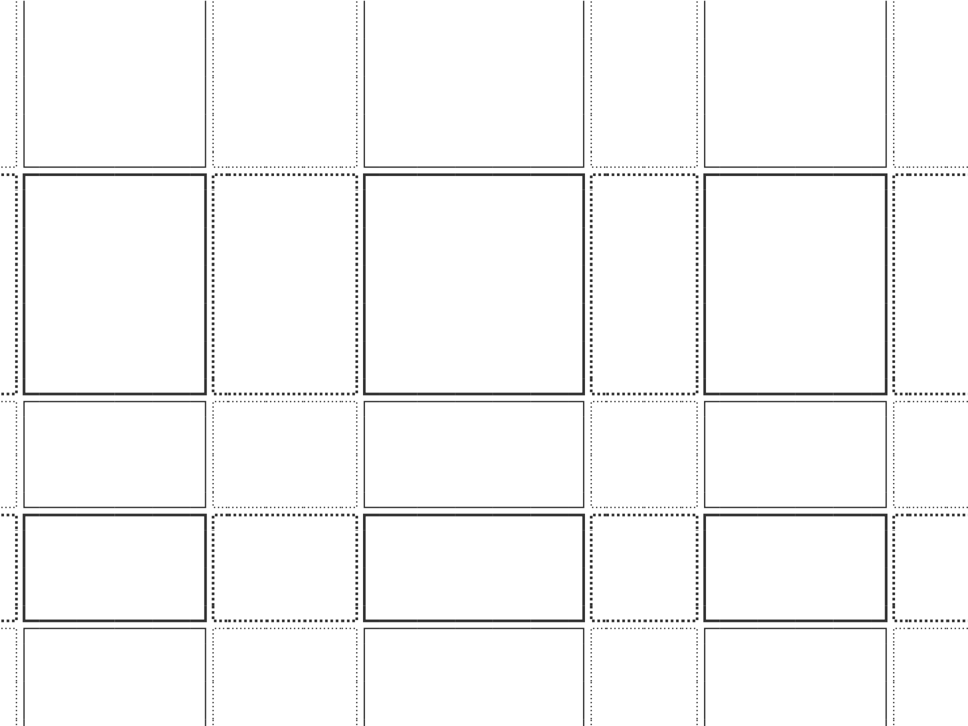
One of the layers of τ_M exactly is the following.



Layer 4: Turing computation (2/4)

We assemble this set τ_M and its three mirror tilesets $\overline{\tau_M}^h$, $\overline{\tau_M}^v$, $\overline{\tau_M}^c$ on a **grid** adding the following tiles to $\tau_M \sqcup \overline{\tau_M}^h \sqcup \overline{\tau_M}^v \sqcup \overline{\tau_M}^c$.





Layer 4: Turing computation (3/4)

We **synchronize** this grid with the one of the first layer.

Lemma. The built tileset is **4-way** deterministic.

Lemma. Every bi-periodic tiling contains a grid on its fourth layer with finite patterns by one of each of the four mirror tilesets of τ_M in each of its four types of rectangles.

It is far from sufficient to **properly simulate** the Turing machine.

Additional restriction 1 (by subset). Horizontal colors containing a **machine state** cannot appear along vertical grid lines on the fourth layer.

This restriction forbids a Turing machine head from **leaving** its rectangle and any other Turing machine head from **appearing** unexpectedly in a rectangle.

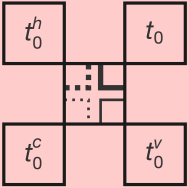
Layer 4: Turing computation (4/4)

We denote as:

- t_0 the **seed tile** of τ_M
- $F \subset \tau_M$ the set of tiles of τ_M carrying a **final state** of M in their north color (*head move* tile)

t_0^h, t_0^v, t_0^c and F^h, F^v, F^c the corresponding objects for the mirror tilesets.

Additional restriction 2 (by grouping). We add the following **initialization** and **halting** constraints on the fourth layer.



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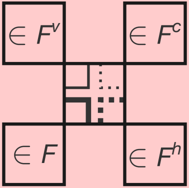
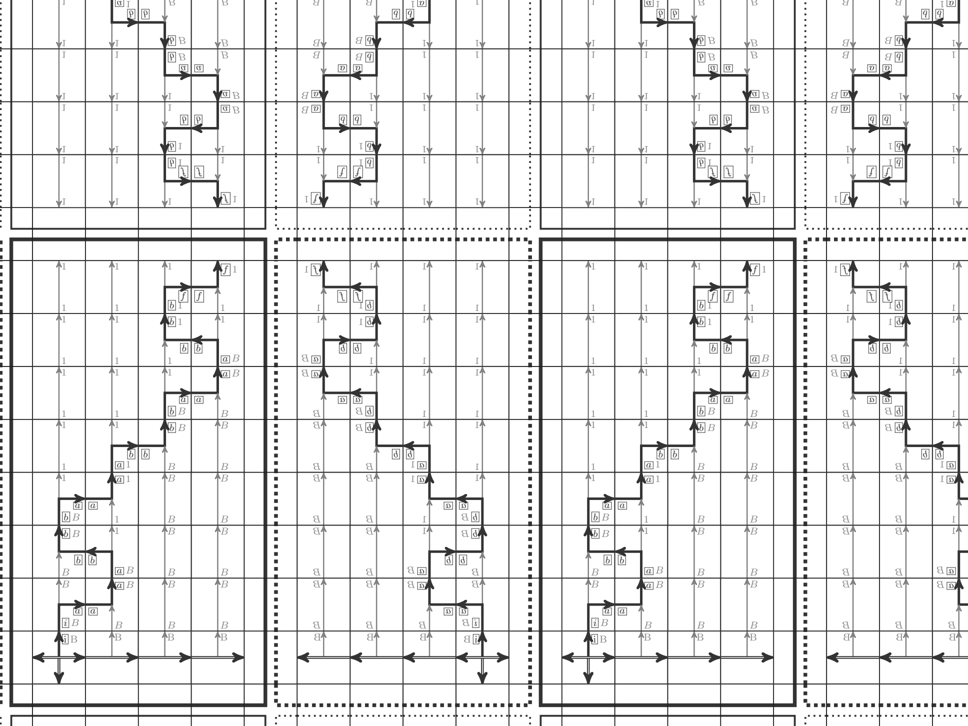


cross D



Conclusion (1/2)

We furthermore require, without loss of generality, *M* to verify the following properties:

- ◇ during the computation, the head never moves to a position situated **to the left** of its starting position;
- ◇ when the computation halts (if it halts), the position of the head is exactly the **rightmost position** that has been reached during the computation.

Lemma. The built tileset is **4-way** deterministic.

Proposition. The built tileset admits periodic tilings $\Leftrightarrow$ the Turing machine *M* halts from the empty tape.

Conclusion (2/2)

Sketch of the proof of the proposition.

$\Leftarrow$ Do to the assumptions on the Turing machine, we can build the previously described **bi-periodic tiling**.

$\Rightarrow$ There must exist a bi-periodic tiling.

Such a tiling contains a grid on its fourth layer with properly initialized (due to the seed tile) Turing computation from the empty tape on the *first* line of each rectangle.

The head cannot leave and no head can appear on the borders, hence the computation is properly simulated.

A final state is required in a corner, thus **the machine halts**.

This concludes the construction, the reduction and the proof of the theorem. ■

4. ↓

That’s all folks!

Thanks for your attention ↓